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On the construction of Hilbert and Quot schemes

Bachelor Thesis

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Abstract

We investigate the classification problems of flat families and flat quotient sheaves on a noetherian projective scheme. In particular, we focus on the questions of representability of the Hilbert, Quot, and related functors.

Chapter one is an introduction to classification problems in Algebraic Geometry and the category-theoretical tools we will need throughout the paper. In chapter two we study a simpler type of Quot functor: the relative Grassmannian. In chapter three we delve deeper into the properties of flat sheaves, which will be crucial in our study of the general Quot functor in chapter four.

The purpose of this paper is expository and none of the main results we present are original.

Contents

Background and Conventions

Our primary reference is Nitin Nitsure's *Contruction of Hilbert and Quot schemes* [Nit05]. On this source is based the overarching theme of this paper, as well as most important theorems. At times we deviate from the approach taken in some proofs, but whenever we do so, a discussion will follow up explaining why the two methods yield equivalent results. For the main topic we also draw inspiration from Dori Bejleri's notes *Moduli Spaces in Algebraic Geometry* [Bej20], which cover most of the same material but under a different light.

As for [Nit05], the prerequisites for this paper are a background in algebraic geometry comparable to chapters 2 and 3 of Hartshorne's textbook [Har77]. Therefore, for all the mathematical notation and conventions we refer to these two sources. Knowledge of basic category theory will be assumed. In this area, Vakil's notes *The Rising Sea* [Vak24] are particularly suited, as they cover all the aspects of functors on schemes that will interest us. Finally, we refer to the *Stacks Project* [Sta24] for the purely technical details which are not available in any of the above sources.

Chapter 1

Classification Problems

1.1 Moduli spaces

Our discussion begins with an expository account of the general philosophy behind classification problems in Algebraic Geometry. As in other mathematical subjects, classifying all objects satisfying a certain property means giving a description of the whole class in terms of invariants, or parameters, which are simpler to understand than the objects themselves. For example, the class of all finitely generated abelian groups is easily classified up to isomorphism in terms of numerical invariants determining the free rank and the torsion subgroups; this information can be encoded in a parameter space which is a subset of $\mathbb{N}^{\mathbb{N}}$.

Oftentimes, the interesting feature of the parameter space is not the underlying set but the additional structure that it carries, which could be algebraic or geometric in nature. For example, the set of all linear isomorphisms of $\mathbb{R}^n$ carries both an $\mathbb{R}$-vector space structure and a smooth manifold structure, given by the Lie group $GL_n(\mathbb{R})$. A less trivial example is the Grassmannian space, which will be discussed in Chapter 2.

In Algebraic Geometry many interesting classes of objects have a parameter space which carries itself an algebro-geometric structure, for example that of a scheme; this is then called the *moduli space* of the classification problem.

Yoneda embedding

In this paper we will cover exclusively classification problems which have a contravariantly functorial nature, meaning that the objects we wish to classify live on schemes and they pull back along morphisms of schemes. This is precisely the behaviour embodied by a contravariant functor $\mathcal{S}ch \to \mathcal{S}et$, and it turns out that a prototype for a vast number of interesting functors is given by the contravariant functor of points.

Category theory gives us a great framework to handle classes of objects and functors in a systematic way. So we will now recall some of the crucial definitions and results that will be employed repeatedly throughout this paper. For a sufficiently detailed account of these results we refer to Vakil's Algebraic Geometry notes [Vak24, §1.3]. Consider an arbitrary category $\mathcal{C}$ for the rest of this section.

Definition 1.1 *For every object $A \in \mathcal{C}$, the* contravariant functor of points *$h_A : \mathcal{C} \to \mathcal{S}et$ associates to each object $B \in \mathcal{C}$ the set $Hom(B, A)$ of all morphisms from B to A and to each morphism $f : C \to B$ the function $g \in Hom(B, A) \mapsto g \circ f \in Hom(C, A)$.*

Definition 1.2 *A contravariant functor $\mathfrak{F} : \mathcal{C} \to \mathcal{S}et$ is called* representable *if there exists an object $F \in \mathcal{C}$ and an isomorphism of functors $\mathcal{N} : h_F \longrightarrow \mathfrak{F}$. We then say that the pair $\langle F, \mathcal{N} \rangle$* represents *$\mathfrak{F}$.*

Lemma 1.3 (Yoneda) *Taking the functor of points gives an equivalence of categories between $\mathcal{C}$ and the subcategory of representable functors in the contravariant functor category $\mathcal{S}et^{\mathcal{C}}$.*

$$h_* : \quad A \in \mathcal{C} \mapsto h_A \in \mathcal{S}et^{\mathcal{C}}$$

In particular, representations of a representable functor are unique, up to a unique isomorphism.

We will search for representations of many different functors corresponding to the classification problem at hand. Then, the moduli space will be the scheme in the representation and the objects of interest living on a scheme T will be determined as the pullbacks along all morphisms from T to the moduli space.

Representability

Let S be a scheme. We denote by $\mathcal{S}ch_S$ the category of S-schemes and by $\langle T, \tau \rangle$ an S-scheme T with structure morphism τ. Whenever the structure morphism is not relevant to the discussion, we try to omit it from the notation. Consider a contravariant functor $\mathfrak{F} : \mathcal{S}ch_S \to \mathcal{S}et$.

Definition 1.4 *The functor $\mathfrak{F}$ is called a* Zariski sheaf *if the following holds for any $T \in \mathcal{S}ch_S$, any open cover $\{U_i\}_i$ of T and any collection $(a_i)_i \in \prod_i \mathfrak{F}(U_i)$:*

> *there exists an element $a \in \mathfrak{F}(T)$ such that $\forall i : \ \iota_i^*(a) = a_i$ if and only if $\forall i, j : \ \iota_{ij}^*(a_i) = \iota_{ji}^*(a_j)$; where the maps ι_i^* and ι_{ij}^* are the pullbacks by the inclusions $\iota_i : U_i \hookrightarrow T$ and $\iota_{ij} : U_i \cap U_j \hookrightarrow U_i$.*

This definition is often abbreviated by saying that for any $T \in \mathcal{S}ch_S$ and any open cover $\{U_i\}_i$ the following is an *equalizer exact sequence*:

$$\mathfrak{F}(T) \to \prod_i \mathfrak{F}(U_i) \rightrightarrows \prod_{i,j} \mathfrak{F}(U_i \cap U_j).$$

Definition 1.5 *A subfunctor $\mathfrak{U} : \mathcal{S}ch_S \to \mathcal{S}et$ of $\mathfrak{F}$ is called* open *(respectively* closed*) if for any $T \in \mathcal{S}ch_S$, the pullback $h_T \times_{\mathfrak{F}} \mathfrak{U}$ is representable by an open (respectively closed) subscheme of T*

Definition 1.6 *A collection of open subfunctors $\{\mathfrak{U}_i\}_i$ of $\mathcal{F}$ is said to* cover *$\mathcal{F}$ if for any $T \in \mathcal{S}ch_S$ and any morphism $f : P \to T$, the open subsets U_i that represent the pullbacks $h_T \times_{\mathfrak{F}} \mathfrak{U}_i$ cover T.*

Unwrapping these definitions yields a criterion to check in practice when subfunctors of a given contravariant functor are open and form a cover. For any $T \in \mathcal{S}ch_S$ and any element $\mathcal{F} \in \mathfrak{F}(T)$ the following has to hold: there exists an open cover $\{U_i\}_i$ of T such that for any i and any morphism $f : P \to T$, f factors through U_i if and only if $f^*\mathcal{F} \in \mathfrak{U}_i(P) \subseteq \mathfrak{F}(P)$.

Lemma 1.7 *The functor $\mathfrak{F}$ is representable if and only if it is a Zariski sheaf and has a cover by representable open subfunctors.*

Proof We refer to [Bej20, 2.10] for the details, but it suffices to verify that the schemes which represent an open cover glue to a scheme which represents the whole functor. □

Universal properties

Let T be a scheme. In this paper we refer to a scheme C with a monomorphism $\iota : C \hookrightarrow T$ as a *subscheme* of T; when the morphism is not relevant we write simply $C \subseteq T$. We will often search for subschemes $C \subseteq T$ satisfying a universal property $\mathfrak{U}$ reading as follows: for every scheme P and every morphism $f : P \to T$, f factors through C if and only if some criterion depending upon the objects P and f is satisfied. So for every P, $\mathfrak{U}$ determines a subset of $Hom(P, T)$ which is contravariantly functorial in P; thus $\mathfrak{U}$ can be thought of as a subfunctor of h_T.

A subscheme $C \subseteq T$ satisfies $\mathfrak{U}$ if and only if it represents $\mathfrak{U}$ as a functor. Moreover, Yoneda's lemma implies that if two subschemes satisfy the same universal property, meaning their functors of points are isomorphic as subfunctors of h_T, then they are uniquely isomorphic as T-schemes, i.e. they are the same subscheme. This is what we mean when we say that a subscheme is uniquely determined by the universal property $\mathfrak{U}$.

Corollary 1.8 *A subfunctor $\mathfrak{U}$ of h_T is representable by a subscheme $\iota : C \hookrightarrow T$ if and only if $\mathfrak{U}$ is a Zariski sheaf and T has an open cover $\{U_i\}_i$ for which the subfunctors $\mathfrak{U} \times_{h_T} h_{U_i}$ of h_{U_i} are representable by subschemes $\iota_i : C_i \hookrightarrow U_i$.*

This is just a rephrasing of the representability criterion 1.7. In addition, the representing subscheme $\iota : C \hookrightarrow T$ is a closed (resp. open, locally closed) embedding if and only if all the subschemes $\iota_i : C_i \hookrightarrow U_i$ are closed (resp. open, locally closed) embeddings, since these properties are local on T. An example of this corollary in action can be found in A.2.

1.2 Basic examples

We will now restate some basic algebraic geometry facts in a way that makes apparent their relation to our discussion above. Let $n \in \mathbb{N}$.

Definition 1.9 *The* global sections functor $\Gamma^n : \mathcal{S}ch \to \mathcal{S}et$ *associates to each scheme* T *the set of* n*-tuples of global sections* $\Gamma(T)^n$ *and to a morphism* $f : P \to T$ *its pullback* $\Gamma(T)^n \to \Gamma(P)^n$ *that acts componentwise.*

Proposition 1.10 *The functor* Γ^n *is representable by affine space* $\mathbb{A}^n_{\mathbb{Z}}$.

Proof For any scheme X, we have that every morphism $X \to \mathbb{A}^n_{\mathbb{Z}}$ is determined by its pullback ringhomomorphism $\Gamma(\mathbb{A}^n_{\mathbb{Z}}) \to \Gamma(X)$ in a functorial way. It is then an elementary algebra fact that ringhomomorphisms from $\mathbb{Z}[x_1, ..., x_n] = \Gamma(\mathbb{A}^n_{\mathbb{Z}})$ are uniquely determined by the choice of n elements in in the codomain. $\square$

Definition 1.11 *The* invertible sheaves functor $\mathfrak{L}^n : \mathcal{S}ch \to \mathcal{S}et$ *associates to each scheme* T *the set*

$$\left\{ (\mathcal{L}, \{\sigma_i\}_{1 \le i \le n}) \;\middle|\; \begin{array}{c} \mathcal{L} \text{ is an invertible sheaf of } \mathcal{O}_T\text{-modules} \\ \{\sigma_i\}_{1 \le i \le n} \text{ are global sections of } T \text{ that generate } \mathcal{L} \end{array} \right\} \Big/ \sim$$

under the equivalence relation of isomorphism consistent with the global sections, and to a morphism $f : P \to T$ *the function* $\langle \mathcal{L}, \{\sigma_i\}_{1 \le i \le n} \rangle \mapsto \langle f^*\mathcal{L}, \{f^*(\sigma_i)\}_{1 \le i \le n} \rangle$. *We denote by* $\langle \mathcal{L}, \{\sigma_i\}_{1 \le i \le n} \rangle$ *an isomorphism class in this set.*

By an isomorphism $\varphi : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ consistent with the respective n-tuples of global sections $\{\sigma_i\}_{1 \le i \le n}$ and $\{\sigma'_i\}_{1 \le i \le n}$, we mean one that satisfies $\varphi(T)(\sigma_i) = \sigma'_i$ for all $1 \le i \le n$.

Proposition 1.12 *The functor* $\mathfrak{L}^{n+1}$ *is representable by projective space* $\mathbb{P}^n_{\mathbb{Z}}$.

Proof This is a standard result proved for example in [Har77, II.7.1], but we will prove it explicitely in greater generality in Section 2.2 $\square$

In view of these results and Yoneda's lemma, affine space $\mathbb{A}^n_{\mathbb{Z}}$ and projective space $\mathbb{P}^n_{\mathbb{Z}}$ are uniquely determined, up to unique isomorphism, as the moduli spaces of the functors Γ^n and $\mathfrak{L}^{n+1}$ respectively.

The above functors also have relative versions. Given a fixed base scheme S, we can generalize them to Γ^n_S and $\mathfrak{L}^{n+1}_S$ acting on the category of S-schemes $\mathcal{S}ch_S$ through the forgetful functor $\mathcal{S}ch_S \to \mathcal{S}ch$. These new functors are again representable, by the S-schemes $\mathbb{A}^n_S$ and $\mathbb{P}^n_S$ respectively.

1.3 Classifying families of subschemes

Let T be a scheme. A *family of subschemes of* $\mathbb{P}^n$ *parameterized by* T will mean a closed subscheme of $\mathbb{P}^n_T$ which is flat over T. The main concern of this paper is classifying all such families and their dependence on T.

Since closed embeddings and flat morphisms are preserved under base extension, this association is contravariantly functorial in T through the fibered product. So we can start our study of the classification problem by defining the corresponding functor.

$$\begin{tikzcd}
Y \times_T P \arrow[rr] \arrow[dr, hook, "\text{closed}"] \arrow[ddr, "\text{flat}"'] & & Y \arrow[dr, hook, "\text{closed}"] \arrow[ddr, "\text{flat}"'] & \\
& \mathbb{P}^n_P \arrow[rr] \arrow[d, "\pi_P"] & & \mathbb{P}^n_T \arrow[d, "\pi_T"] \\
& P \arrow[rr, "f"] & & T
\end{tikzcd}$$

Definition 1.13 *The* Hilbert functor $\mathfrak{Hilb}_{\mathbb{P}^n} : \mathcal{S}ch \to \mathcal{S}et$ *associates to each scheme T the set*

$$\{Y \subseteq \mathbb{P}^n_T \mid Y \textit{ is a family of subschemes of } \mathbb{P}^n \textit{ parameterized by } T\},$$

and to each morphism $P \to T$ the function $Y \mapsto Y \times_T P$.

Recall that for any scheme S, there is a one to one correspondence between its closed subschemes $Y \subseteq S$ and its quasi-coherent quotient sheaves $\mathcal{O}_S \twoheadrightarrow \mathcal{F}$ of $\mathcal{O}_S$-modules, given by the function $Y \mapsto \iota_* \mathcal{O}_Y$ [Har77, II.5.9]. Since the flatness of a closed subscheme Y is expressed in terms of $\mathcal{O}_Y$, this suggests that we can rewrite our functor $\mathfrak{Hilb}_{\mathbb{P}^n}$ in terms of quotient sheaves. The obvious contravariant relation between sheaves of modules on schemes is the pullback functor, which is right exact and thus sends quotient sheaves to quotient sheaves. In addition, we verify in the appendix A.1 that given a morphism $f : P \to T$ and a closed subscheme $Y \subseteq T$, the quotient sheaf of $Y \times_T P$ on P is the pullback by f of the quotient sheaf of Y on T. Thus, the correspondence respects the contravariant relation of our functor. This allows us to rewrite our classification problem by defining the following functor and showing that it is isomorphic to $\mathfrak{Hilb}_{\mathbb{P}^n}$.

Definition 1.14 *The* family of quotients functor $\mathfrak{Quot}_{\mathbb{P}^n} : \mathcal{S}ch \to \mathcal{S}et$ *associates to each scheme T the set*

$$\left\{ (\mathcal{F}, q) \;\middle|\; \begin{array}{c} \mathcal{F} \textit{ is a quasi-coherent } \mathcal{O}_{\mathbb{P}^n_T}\textit{-module flat over } T \\ q : \mathcal{O}_{\mathbb{P}^n_T} \twoheadrightarrow \mathcal{F} \end{array} \right\} \Big/ \sim$$

under the equivalence relation of isomorphism as quotient sheaves of $\mathcal{O}_{\mathbb{P}^n_T}$, and to each morphism $f : P \to T$ the function $\langle \mathcal{F}, q\rangle \mapsto \langle f_{\mathbb{P}}^* \mathcal{F}, f_{\mathbb{P}}^* q\rangle$; where $f_{\mathbb{P}}$ denotes the base extension of f by $\pi_T : \mathbb{P}^n_T \to T$..

Here and in the rest of this paper, all sheaves on a scheme T and their respective morphisms are understood to be sheaves of $\mathcal{O}_T$-modules with $\mathcal{O}_T$-linear homomorphisms.

Lemma 1.15 *There is an isomorphism of functors $\mathfrak{Hilb}_{\mathbb{P}^n} \cong \mathfrak{Quot}_{\mathbb{P}^n}$.*

Proof As described above, we define a natural transformation $\mathcal{N}$ that acts on a scheme T by the function $Y \in \mathfrak{Hilb}_{\mathbb{P}^n}(T) \mapsto \langle \iota_* \mathcal{O}_Y, \iota^\# \rangle \in \mathfrak{Quot}_{\mathbb{P}^n}(T)$, where ι is the inclusion $Y \hookrightarrow T$. This is the same transformation that relates the two more general isomorphic functors handled in A.1. In fact, $\mathfrak{Hilb}_{\mathbb{P}^n}$ and $\mathfrak{Quot}_{\mathbb{P}^n}$ fit in the following diagram as subfunctors of $\mathcal{S}ubs$ and $\mathcal{Q}coh$ respectively, using the same notation as in the appendix.

$$\begin{tikzcd} \mathfrak{Hilb}_{\mathbb{P}^n} \arrow[r, "\mathcal{N}"] \arrow[d, hook] & \mathfrak{Quot}_{\mathbb{P}^n} \arrow[d, hook] \\ \mathcal{S}ubs \arrow[r, "\mathcal{N}"] & \mathcal{Q}coh \end{tikzcd}$$

Thus it only remains to check that, for any scheme T, the restriction of $\mathcal{N}$ to $\mathfrak{Hilb}_{\mathbb{P}^n}(T)$ maps bijectively into $\mathfrak{Quot}_{\mathbb{P}^n}(T)$.
Indeed, a closed subscheme $Y \subseteq \mathbb{P}^n_T$ is flat over T if and only if, for every $y \in Y$, $(\iota_* \mathcal{O}_Y)_{y, \mathbb{P}^n_T} = \mathcal{O}_{y,Y}$ is a flat $\mathcal{O}_{\pi(y),T}$-module through the ring homomorphism $(\pi \circ \iota)^\#_{\pi(y)}$. Since $Y = Supp\, \iota_* \mathcal{O}_Y$, this holds if and only if the sheaf $\iota_* \mathcal{O}_Y$, which is the corresponding quotient sheaf $\mathcal{F}$, is flat over T. Thus there is a bijection between the subfunctors. $\square$

By restricting the functor $\mathfrak{Quot}_{\mathbb{P}^n}$ to the subcategory of locally noetherian schemes, denoted $\underline{\mathcal{S}ch}$, we obtain that all quasi-coherent quotient sheaves of the coherent sheaf $\mathcal{O}_{\mathbb{P}^n_T}$ are also coherent. This restriction will enable us to prove that our restricted functor is representable by a locally noetherian scheme $Quot_{\mathbb{P}^n}$. In order to do so, we will set aside temporarily the study of flat families to concentrate first on a similar functor, which classifies a specific type of coherent quotient sheaves: locally free sheaves of finite fixed rank, i.e. vector bundles.

Chapter 2

Grassmannians

Let $r \geq d \in \mathbb{N}$ for the rest of this chapter.

2.1 The Grassmannian functor

We start by recalling the construction of the Grassmannian space in classical topology. Given an r dimensional real vector space V we can consider the set of its d dimensional subspaces and topologize it in a meaningful way. Namely, any two subspaces are related by a $d \times r$ matrix of full rank and in a unique way up to postmultiplication with an invertible $d \times d$ matrix; this is done by choosing d basis vectors for both subspaces and completing them *in the same way* to bases of V. Thus, the subset of all subspaces whose bases can be completed *in the same way* can be parameterized by $dr - d^2 = d(r-d)$ real numbers, thereby importing the topology from $\mathbb{R}^{d(r-d)}$. These $\binom{r}{d}$ different subsets, called *Schubert Cells*, are enough to form an open cover of the whole set we were interested in. This is the explicit presentation of the Grassmannian $Grass(V, d)$ and a detailed construction can be found for example in [Vak24, §7.7].

In addition to the topology, $Grass(V, d)$ also carries a smooth manifold structure, imported along with the topology from $\mathbb{R}^{d(r-d)}$ or by characterizing it as a quotient of the Lie Group $GL(V)$. In the context of Section 1.1, we can say that the set $Grass(V, d)$ provides insight into the classification problem of d dimensional subspaces of V through its additional structures.

By passing from vector spaces to vector bundles over schemes, we find an analogous classification problem in Algebraic Geometry. Namely, given a scheme T, we wish to describe the set

$$\left\{ (\mathcal{K}, \kappa) \;\middle|\; \begin{array}{c} \mathcal{K} \text{ is a rank d vector bundle on T} \\ \kappa : \mathcal{K} \hookrightarrow \mathcal{O}_T^{\oplus r} \end{array} \right\} \Big/ \sim$$

under the equivalence relation of *isomorphism as subsheaves of* $\mathcal{O}_T^{\oplus r}$. We call a class in this set a *rank d subbundle of* $\mathcal{O}_T^{\oplus r}$ and denote it by $\langle \mathcal{K}, \kappa \rangle$.

This association is almost, but not quite, contravariantly functorial in T through the pullback, as structure sheaves get pulled back to structure sheaves and likewise locally free sheaves of modules. But the pullback functor itself is in general only right exact, meaning that the inclusion might not be pulled back to an injective morphism of sheaves.

By restricting attention to subbundles whose inclusion in $\mathcal{O}_T^{\oplus r}$ has cokernel that is likewise locally free, and by rewriting the subbundles in the form of a SES, we get the set

$$\left\{ (0 \to \mathcal{K} \xrightarrow{\kappa} \mathcal{O}_T^{\oplus r} \xrightarrow{q} \mathcal{F} \to 0) \;\middle|\; \begin{array}{c} \mathcal{K}, \mathcal{F} \text{ are vector bundles on T} \\ \text{of rank } d \text{ and } r-d \text{ respectively} \end{array} \right\} \Big/ \sim$$

under the equivalence relation of *isomorphism as short exact sequences*. We can exploit the right exactness of the pullback by applying it on the cokernel to obtain a contravariant relation in T: given a morphism $f : P \to T$, each SES in the above set is mapped to $(0 \to ker(f^*q) \to \mathcal{O}_P^{\oplus r} \to f^*\mathcal{F} \to 0)$, where $ker(f^*q)$ is again a rank d vector bundle. Obviously this mapping respects the equivalence relation and commutes with compositions of morphisms. We can rewrite again this set by omitting the redundant first term in the SES and switching the roles of d and $r-d$ for notational convenience, as will become apparent in the future.

Definition 2.1 *The* Grassmannian functor $\mathfrak{Grass}(r, d) : \mathcal{S}ch \to \mathcal{S}et$ *associates to each scheme T the set*

$$\left\{ (\mathcal{F}, q) \;\middle|\; \begin{array}{c} \mathcal{F} \textit{ is a rank d vector bundle on T} \\ q : \mathcal{O}_T^{\oplus r} \twoheadrightarrow \mathcal{F} \end{array} \right\} \Big/ \sim$$

under the equivalence relation of isomorphism as quotient sheaves of $\mathcal{O}_T^{\oplus r}$, and to each morphism $f : P \to T$ the function $\langle \mathcal{F}, q \rangle \mapsto \langle f^*\mathcal{F}, f^*q \rangle$. We call a class in this set a rank d quotient bundle of $\mathcal{O}_T^{\oplus r}$ *and denote it by $\langle \mathcal{F}, q \rangle$.*

In conclusion, the classification problem of rank $r-d$ subbundles with locally free cokernel, which is equivalent to the one of rank d quotient bundles, assumes a functorial nature embodied by $\mathfrak{Grass}(r, d)$.

2.2 The Grassmannian scheme and its properties

In this section we will prove that the Grassmannian functor $\mathfrak{Grass}(r,d)$ has a representing object $Grass(r,d)$ in the category of schemes. Then we will study some of its properties.

The *Grassmannian scheme* $Grass(r,d)$ can be explicitely constructed as the gluing of $\binom{r}{d}$ affine patches $\mathbb{A}_{\mathbb{Z}}^{d(r-d)}$, again called *Schubert Cells*, through transition maps which mirror the action of $d \times d$ invertible matrices in the construction of the classical Grassmannian outlined above. There is a rank d quotient bundle $\mathcal{O}_{Grass(r,d)}^{\oplus r} \twoheadrightarrow \mathcal{Q}$, called the *tautological bundle* , with the property that its pullbacks by the morphisms $T \to Grass(r,d)$ from a scheme T determine uniquely all the rank d quotient bundles of $\mathcal{O}_T^{\oplus r}$. In addition, this correspondence is natural, meaning that the functor of points $h_{Grass(r,d)}$ and $\mathfrak{Grass}(r,d)$ are isomorphic as functors through taking the pullback of the tautological bundle. This is the approach taken in our primary reference [Nit05].

However, in this paper we follow a different approach, outlined in [Vak24, 16.4] and carried through in [Bej20, 3.1]. This is done in the interest of illustrating a classical application of the category-theoretical machinery that we introduced in Section 1.1. Namely, we will use the representability criterion 1.7.

In both approaches it is possible to see how the Grassmannian is a generalization of projective space. In the first, the explicit construction of $Grass(r,1)$ with its tautological bundle is precisely the one of $Proj(\mathbb{Z}[x_1,...,x_r]) = \mathbb{P}_{\mathbb{Z}}^{r-1}$ with its twisting sheaf $\mathcal{O}(1)$. In the second, we observe how the functor $\mathfrak{Grass}(r,1)$ is isomorphic to the invertible sheaves functor $\mathfrak{L}^r$ seen in 1.11; namely, rank 1 quotient bundles of $\mathcal{O}_T^{\oplus r}$ are in natural correspondence with invertible sheaves $\mathcal{L}$ on T with a choice of r generating global sections. Therefore, Yoneda's lemma 1.3 implies that their moduli spaces are uniquely isomorphic.

Theorem 2.2 *The functor $\mathfrak{Grass}(r,d)$ is representable by a finite type scheme $Grass(r,d)$ over $\mathbb{Z}$.*

Proof First we prove that $\mathfrak{Grass}(r,d)$ is a Zariski sheaf. Let T be any scheme and $\{U_i\}_i$ an open cover of T. We have to show exactness of the equalizer sequence

$$\mathfrak{Grass}(r,d)(T) \to \prod_i \mathfrak{Grass}(r,d)(U_i) \rightrightarrows \prod_{i,j} \mathfrak{Grass}(r,d)(U_i \cap U_j). \tag{2.1}$$

So let $\{\langle \mathcal{F}_i, q_i \rangle\}_i$ be an arbitrary collection of elements in each of the sets $\mathfrak{Grass}(r,d)(U_i)$. The maps of sets in (2.1) are the pullback of quotient bundles by the inclusion morphisms of schemes, i.e. their restriction to open subsets. Therefore, if $\langle \mathcal{F}_i|U_j, q_i|U_j\rangle = \langle \mathcal{F}_j|U_i, q_j|U_i\rangle$ in $\mathfrak{Grass}(r,d)(U_i \cap U_j)$ for every i and j, then by the definition of the equivalence relation there are isomorphisms of quotient sheaves $\varphi_{ij} : \mathcal{F}_i|U_j \to \mathcal{F}_j|U_i$ on $U_i \cap U_j$. These satisfy the cocycle condition and thus glue to a unique global quotient bundle $\langle \mathcal{F}, q\rangle$ whose restriction on U_i is precisely $\langle \mathcal{F}_i, q_i\rangle$ for every i.

Now we define a subfunctor of $\mathfrak{Grass}(r,d)$ for each subset of $\{1,...,r\}$ with d elements, considered as an ordered d-tuple, and afterwards we prove that they form an open cover. So let I be any d-tuple. Define the I^{th} Grassmannian functor $\mathfrak{Grass}(r,d)_I$ that associates to each scheme T the subset

$$\{\langle \mathcal{F}, q\rangle \in \mathfrak{Grass}(r,d)(T) \,|\, q \circ \iota_I : \oplus^d \mathcal{O}_S \twoheadrightarrow \mathcal{F}\},$$

where ι_I denotes the inclusion morphism $\mathcal{O}_T^{\oplus d} \hookrightarrow \mathcal{O}_T^{\oplus r}$ which maps the i^{th} summand to the I_i^{th} summand. Showing that $\mathfrak{Grass}(r,d)_I$ is an open subfunctor amounts to finding for any scheme T and any quotient bundle $\langle \mathcal{F}, q\rangle \in \mathfrak{Grass}(r,d)(T)$ an open subset $U_I \subset T$ such that for every morphism $f : P \to T$, f factors through U_I if and only if $f^*\langle \mathcal{F}, q\rangle \in \mathfrak{Grass}(r,d)_I(P)$.

Let $p \in P$ and $t = f(p) \in T$. By Nakayama's lemma, the $\mathcal{O}_{t,T}$-module homomorphism $(q \circ \iota_I)_t$ is surjective if and only if

$$\overline{(q \circ \iota_I)_t} : k(t)^d \to k(t)^r \to \mathcal{F}_t / m_t \mathcal{F}_t, \tag{2.2}$$

its quotient by the maximal ideal $m_t \subset \mathcal{O}_{t,T}$, is surjective. The same holds for $(f^*q \circ f^*\iota_I)_p$ and

$$\overline{(f^*q \circ f^*\iota_I)_p} : k(p)^d \to k(p)^r \to (f^*\mathcal{F})_p / m_p (f^*\mathcal{F})_p. \tag{2.3}$$

Since 2.2 and 2.3 are linear maps of vector fields, and the latter is obtained from the former by extension of scalars through the field extension $\overline{f_p^\#} : k(t) \to k(p)$, one is surjective if and only if the other is. So define U_I to be the subset of points $t \in T$ where $(q \circ \iota_I)_t$ is surjective. By the chain of equivalences above, $f(p) \in U_I$ for all $p \in P$ if and only if $f^*\langle \mathcal{F}, q\rangle \in \mathfrak{Grass}(r,d)_I(P)$. U_I is indeed open because its complement $Supp(coker(q \circ \iota_I))$ is closed, as $q \circ \iota_I$ is a morphism of finite type quasi-coherent sheaves. Since T and $\langle \mathcal{F}, q\rangle$ were arbitrary $\mathfrak{Grass}(r,d)_I$ is an open subfunctor.

Moreover, for any T and $\langle \mathcal{F}, q\rangle$ as above, the open sets U_I cover T as I ranges over all the d-tuples. Indeed, for any $t \in T$, the linear map of $k(t)$-vector spaces $\overline{q_t}$ is surjective so there must exist at least d out of the r summands that span the d dimensional vector space $\mathcal{F}_t / m_t \mathcal{F}_t$. This is enough to show that the open subfunctors above cover $\mathfrak{Grass}(r,d)$.

Finally, given any d-tuple I, we show that $\mathfrak{Grass}(r,d)_I$ is representable. For any scheme T, every quotient bundle $\langle \mathcal{F}, q\rangle \in \mathfrak{Grass}(r,d)_I(T)$ turns out to be just the free sheaf $\mathcal{O}_{\mathcal{O}_T}^{\oplus d}$ because $q \circ \iota_I$ is an isomorphism on trivializing neighbourhoods of $\mathcal{F}$ which cover T, so an isomorphism on all of T. Note that split exact sequences

$$0 \to \mathcal{O}_T^{\oplus d} \xrightarrow{\iota_I} \mathcal{O}_T^{\oplus r} \xrightarrow{q} \mathcal{F} \to 0$$

are uniquely determined by a choice of d global sections for each of the complementary $n-d$ components of the central term, i.e. by $d(r-d)$ global sections. Thus, after checking that this correspondence is natural, we recognize $\mathfrak{G}rass(r,d)_I$ as the global sections functor $\Gamma^{d(r-d)}$, so we conclude with 1.10 that $Grass(r,d)_I \cong \mathbb{A}^d_{(r-d)}$.

The resulting scheme $Grass(r,d)$ is of finite type over $\mathbb{Z}$ because it is the gluing of finitely many affine schemes $\mathbb{A}_{\mathbb{Z}}^{d(r-d)}$ which are all of finite type.□

Having proved that $\mathfrak{G}rass(r,d)$ is representable, we also obtain by Yoneda's lemma the tautological bundle on $Grass(r,d)$, mentioned at the beginning of this section. Namely, it is the unique object $\langle \mathcal{Q}, q\rangle \in \mathfrak{G}rass(r,d)(Grass(r,d))$, so a rank d quotient bundle of $\mathcal{O}^{\oplus r}_{Grass(r,d)}$, which determines for every scheme T the natural isomorphism

$$f \in h_{Grass(r,d)}(T) \mapsto \langle f^*\mathcal{Q}, f^*q\rangle \in \mathfrak{G}rass(r,d)(T). \tag{2.4}$$

In the particular case of $Grass(r,1)$, we recognize the tautological bundle as the twisting sheaf with the r coordinate functions on $\mathbb{P}^{r-1}_{\mathbb{Z}}$. This is because $\langle \mathcal{O}(1), \{x_i\}_{0\le i\le r-1}\rangle \in \mathfrak{L}^r(\mathbb{P}^{r-1}_{\mathbb{Z}})$ is the unique invertible sheaf that represents the isomorphism of functors $h_{\mathbb{P}^{r-1}_{\mathbb{Z}}} \xrightarrow{\sim} \mathfrak{L}^r$ mentioned in 1.12.

Proposition 2.3 *$Grass(r,d)$ is proper over $\mathbb{Z}$.*

$Grass(r,d)$ is of finite type over $\mathbb{Z}$, so we can use the valuative criterion for properness [Har77, II.4.7]. This is particularly helpful when checking properness of a moduli space because the valuative criterion can be phrased completely in terms of the corresponding functor of points, without having to deal with the underlying scheme.

Proof Let R be a DVR and K its function field. We have to check that for any commutative diagram

$$\begin{tikzcd} Spec\,K \arrow[r] \arrow[d] & Grass(r,d) \arrow[d] \\ Spec\,R \arrow[r] \arrow[ur, dashed] & Spec\,\mathbb{Z} \end{tikzcd}$$

there exists a unique dashed arrow. Since $Grass(r,d)$ represents the functor $\mathfrak{G}rass(r,d)$, this is equivalent to the restriction map

$$\mathfrak{G}rass(r,d)(Spec\,R) \to \mathfrak{G}rass(r,d)(Spec\,K)$$

being bijective. Let $\langle\mathcal{F}, q\rangle \in \mathfrak{Grass}(r,d)(Spec\,K)$. The condition that a rank d quotient bundle $p : \mathcal{O}_R^{\oplus r} \twoheadrightarrow \mathcal{M}$ on $Spec\,R$, with R-module homomorphism $p : R^r \twoheadrightarrow M$, gets pulled back to $q : \mathcal{O}_K^{\oplus r} \twoheadrightarrow \mathcal{F}$, with K-module homomorphism $q : K^r \twoheadrightarrow F$, translates to F and $M \otimes_R K$ being isomorphic as quotient modules of K^r. This condition is satisfied by the image of R^r in F under q. Furthermore, for two such quotient modules $R^r \twoheadrightarrow M, M'$, any K-isomorphism $M \otimes_R K \xrightarrow{\sim} F \xrightarrow{\sim} M' \otimes_R K$ of quotient modules of K^r restricts to an R-isomorphism $M \xrightarrow{\sim} M'$ of quotient modules of R^r.

$$\begin{tikzcd}
Spec\,K \arrow[r, squiggly, mapsto] \arrow[dd, hook, "j"] & K^r \arrow[rr, two heads, "q"] \arrow[dr, two heads, "p\otimes K"] & & F & K\text{-modules} \\
& & M \otimes_R K \arrow[ur, dashed, "\sim"'] & & \\
Spec\,R \arrow[r, squiggly, mapsto] & R^r \arrow[uu, hook] \arrow[r, two heads, "p"] & M \arrow[u, hook] & & R\text{-modules}
\end{tikzcd}$$

So we recognize the image sheaf $p : \mathcal{O}_R^{\oplus r} \twoheadrightarrow \mathcal{M}$ of the composition $\mathcal{O}_R^{\oplus r} \to j_*\mathcal{O}_K^{\oplus r} \twoheadrightarrow j_*\mathcal{F}$ as the unique sheaf on $Spec\,R$, up to isomorphism as quotient sheaves of $\mathcal{O}_R^{\oplus r}$, pulling back to $\langle\mathcal{F}, q\rangle$. In addition, $\mathcal{M}$ is a vector bundle because its R-module M is free, being an R-submodule of the K-vector space F. Thus, we have verified the valuative criterion. $\square$

Proposition 2.4 *The d^{th} alternating power Λ^d induces a closed embedding of functors $\mathfrak{Grass}(r,d) \hookrightarrow \mathfrak{Grass}(\binom{r}{d}, 1)$. As a consequence, there is a closed embedding of schemes $Grass(r,d) \hookrightarrow \mathbb{P}_{\mathbb{Z}}^{\binom{r}{d}-1}$.*

Note that every morphism from $Grass(r,d)$ to $\mathbb{P}^n_{\mathbb{Z}}$ is proper, because these are both proper schemes over $\mathbb{Z}$ [Sta24, 01W6]. Also recall that a proper monomorphism is a closed embedding [Sta24, 04XV], so for the proof it is enough to show that Λ^d induces a monomorphism into $\mathbb{P}_{\mathbb{Z}}^{\binom{r}{d}-1}$. This type of argument will be recurring when searching for a closed embedding.

Proof We show that Λ^d expresses $\mathfrak{Grass}(r,d)$ as a subfunctor of $\mathbb{P}_{\mathbb{Z}}^{\binom{r}{d}-1} = \mathfrak{Grass}(\binom{r}{d}, 1)$. The corresponding natural transformation between the functors of points $h_{Grass(r,d)}$ and $h_{Grass(\binom{r}{d},1)}$ will then yield by Yoneda's lemma a monomorphism of schemes $\mathcal{P} : Grass(r,d) \to Grass(\binom{r}{d}, 1)$, thus a closed embedding by the remark above.

So define a natural transformation $\mathcal{N}$, which on any scheme T acts as

$$\langle\mathcal{F}, q\rangle \in \mathfrak{Grass}(r,d)(T) \mapsto \langle\Lambda^d\mathcal{F}, \Lambda^d q\rangle \in \mathfrak{Grass}(\tbinom{r}{d}, 1)(T), \tag{2.5}$$

This map is well defined because $\Lambda^d\mathcal{O}_T^{\oplus r} \cong \mathcal{O}_T^{\oplus\binom{r}{d}}$, $\Lambda^d\mathcal{F}$ is a rank 1 vector bundle and the alternating power is a right exact functor. It is natural because the pullback of vector bundles commutes with the alternating power.

For convenience, let us index the summands of $\mathcal{O}_T^{\oplus\binom{r}{d}}$ by the $\binom{r}{d}$ different d-tuples so that for each I, $\Lambda^d(\iota_I) : \mathcal{O}_T^{\oplus 1} = \Lambda^d\mathcal{O}_T^{\oplus d} \hookrightarrow \Lambda^d\mathcal{O}_T^{\oplus r}$ becomes precisely $\mathcal{O}_T \xhookrightarrow{\iota_I} \mathcal{O}_T^{\oplus\binom{r}{d}}$ under the isomorphisms mentioned above. In doing so, we get that the restriction of $\mathcal{N}$ to the open subfunctor $\mathfrak{G}rass(r,d)_I$ maps to the open subfunctor $\mathfrak{G}rass(\binom{r}{d},1)_I$. So now it is enough to prove that all the restrictions $\mathcal{N}_I$ express a subfunctor, implying that $\mathcal{N}$ also will.

As we have seen in the proof of the representability of $\mathfrak{G}rass(r,d)_I$, for any scheme T, this functor is the choosing of $d(r-d)$ global sections which complete a $d \times r$ matrix with the identity matrix as its I^{th} $d \times d$ submatrix. Through this correspondence, $\mathcal{N}_I(T)$ maps all the $d \times d$ minors of the matrix to their spot in the $1 \times \binom{r}{d}$ matrix. It is now a pure commutative algebra exercise to prove that our $d \times r$ matrix over the ring of global sections is uniquely determined by all of its $d \times d$ minors, i.e. $\mathcal{N}_I(T)$ is an injective map. Thus $\mathcal{N}_I$ expresses $\mathfrak{G}rass(r,d)_I$ as a subfunctor of $\mathfrak{G}rass(\binom{r}{d},1)$. $\square$

The morphism $\mathcal{P}$ we obtained into projective space is called the *Plücker embedding*. We recognize the determinant of the tautological bundle $\Lambda^d q : \Lambda^d\mathcal{O}^{\oplus r}_{Grass(r,d)} \twoheadrightarrow \Lambda^d\mathcal{Q}$ on $Grass(r,d)$ as the pullback by $\mathcal{P}$ of the twisting sheaf $\mathcal{O}(1)$ on $\mathbb{P}_{\mathbb{Z}}^{\binom{r}{d}-1}$. Under this pullback, the $\binom{r}{d}$ coordinate functions x_I that generate $\mathcal{O}(1)$ are sent to the global sections of $\Lambda^d\mathcal{Q}$ determined by the summands of $\Lambda^d\mathcal{O}^{\oplus r}_{Grass(r,d)} \cong \mathcal{O}^{\oplus\binom{r}{d}}_{Grass(r,d)}$; they are called *Plücker coordinates*.

It is worth noting that, had we proved representability by explicitely constructing $Grass(r,d)$ and its tautological bundle Q, we could have defined the Plücker embedding in terms of the Plücker coordinates and the determinant bundle $\Lambda^d\mathcal{Q}$, as in [Nit05]. So the two approaches are truly complementary in the information they give us about the scheme $Grass(r,d)$.

One could try to encapsulate this last consideration into a metamathematical commutative diagramm . . .

$$\begin{tikzcd}
\prod_I \mathbb{A}_{\mathbb{Z}}^{d(r-d)} \arrow[r, "\sim"] \arrow[d, "h_*"] & \prod_I Grass(r,d)_I \arrow[r, "\text{gluing schemes}"] \arrow[d, "h_*"] & Grass(r,d) \arrow[d, "h_*"] \\
\prod_I \Gamma(\,\cdot\,, \mathcal{O}^{\oplus d(r-d)}) \arrow[r, "\sim"] & \prod_I \mathfrak{G}rass(r,d)_I \arrow[r, "\text{gluing functors}"] & \mathfrak{G}rass(r,d)
\end{tikzcd}$$

. . . but lets not try to exacerbate the level of abstract nonsense.

2.3 Relative Grassmannians

In this section we will generalize to the *relative Grassmannian functor* and then prove its representability. Namely, we want to classify rank d quotient bundles not just of the free sheaf $\mathcal{O}_T^{\oplus r}$ but of any coherent sheaf $\mathcal{E}$ on T. To approach the problem in a functorial way as before, this $\mathcal{E}$ cannot depend on T but should be fixed beforehand. So we fix a base scheme S and a coherent sheaf $\mathcal{E}$ on it for the rest of this section. Then, by restricting to the category of S-schemes we can study quotient bundles of the pullback of $\mathcal{E}$ along the structure morphism of T in a contravariantly functorial way.

Definition 2.5 *The* relative Grassmannian functor $\mathfrak{G}rass_S(\mathcal{E}, d) : \mathcal{S}ch_S \to \mathcal{S}et$ *associates to each S-scheme* $\langle T, \tau \rangle$ *the set*

$$\left\{ (\mathcal{F}, q) \;\middle|\; \begin{array}{c} \mathcal{F} \text{ is a rank } d \text{ vector bundle on } T \\ q : \mathcal{E}_T \twoheadrightarrow \mathcal{F} \end{array} \right\} \Big/ \sim$$

under the equivalence relation of isomorphism as quotient sheaves of $\mathcal{E}_T$*; where with* $\mathcal{E}_T$ *is meant the pullback* $\tau^*\mathcal{E}$*.*

$$\begin{tikzcd}
\mathcal{E}_P \arrow[r, two heads] \arrow[d, squiggly] & f^*\mathcal{F} \arrow[dl, squiggly] & \mathcal{E}_T \arrow[r, two heads] \arrow[d, squiggly] & \mathcal{F} \arrow[dl, squiggly] \\
P \arrow[rr, "f"] \arrow[drrr, dotted] & & T \arrow[dr, dotted, "\tau"] & \mathcal{E} \arrow[d, squiggly] \\
& & & S
\end{tikzcd}$$

For any morphism $\sigma : S' \to S$, we have $\mathfrak{G}rass_{S'}(\sigma^*\mathcal{E}, d) = \mathfrak{G}rass_S(\mathcal{E}, d) \circ F_\sigma$, where F_σ is the postcomposition of structure morphisms $\mathcal{S}ch_{S'} \to \mathcal{S}ch_S$. Out of this remark follows a useful base change property of our functor.

Lemma 2.6 *If* $\mathfrak{G}rass_S(\mathcal{E}, d)$ *is representable by an S-scheme G, then* $\mathfrak{G}rass_{S'}(\sigma^*\mathcal{E}, d)$ *is representable by the* S'*-scheme* $G \times_S S'$*.*

Proof This follows from Yoneda's lemma, because there is an isomorphism of functors of S'-schemes $h_{G \times_S S'} \to \mathfrak{G}rass_{S'}(\sigma^*\mathcal{E}, d)$. On an S'-scheme $\langle T, \tau \rangle$ this natural transformation acts by postcomposition with the projection $G \times_S S' \to G$:

$$Hom_{S'}(\langle T, \tau \rangle, G \times_S S') \xrightarrow{\sim} Hom_S(\langle T, \sigma \circ \tau \rangle, G) = (\mathfrak{G}rass_S(\mathcal{E}, d) \circ F_\sigma)(T)$$

$$f \mapsto \pi_G \circ f$$

This is a bijection because of the universal property of the fiber product. □

Note that $\mathfrak{Grass}_{\mathbb{Z}}(\mathcal{O}_{\mathbb{Z}}^{\oplus r}, d) = \mathfrak{Grass}(r, d)$, and by the previous lemma we also have that $\mathfrak{Grass}_S(\mathcal{O}_S^{\oplus r}, d)$ is representable by the S-scheme $Grass(r, d) \times S$. However, the base change property doesn't help us further in the quest for representability because not every coherent sheaf $\mathcal{E}'$ on S' is the pullback of a coherent sheaf $\mathcal{E}$ on S.

The following lemma will play a fundamental role in our study of the relationships between different Grassmannians throughout the paper. For the time being, we will use it to prove representability of $\mathfrak{Grass}_S(\mathcal{E}, d)$. Just as it is sketched in [Nit05, 1.(2)], the lemma allows us, at least locally, to reduce to the case that $\mathcal{E}$ is globally generated.

Lemma 2.7 *For any surjection of coherent sheaves $\sigma : \mathcal{E}' \twoheadrightarrow \mathcal{E}$ on S, precomposition by σ induces a closed embedding of functors $\mathfrak{Grass}_S(\mathcal{E}, d) \hookrightarrow \mathfrak{Grass}_S(\mathcal{E}', d)$.*

Proof Let $T \in \mathcal{S}ch_S$. Denote by $\sigma_T : \mathcal{E}'_T \twoheadrightarrow \mathcal{E}_T$ the base extension of σ by τ, which is still a surjection of coherent sheaves. Precomposition by σ_T is a natural transformation in T which respects the equivalence relation in $\mathfrak{Grass}_S(\mathcal{E}, d)(T)$ and leaves invariant the rank d. It is injective because an isomorphism as quotient bundles of $\mathcal{E}'$ is also one as quotients of $\mathcal{E}$, so we have to verify that it expresses a closed subfunctor. Let $\langle \mathcal{F}, q \rangle \in \mathfrak{Grass}_S(\mathcal{E}', d)(T)$, and consider a morphism of S-scheme $f : P \to T$.

$$\begin{tikzcd} f^*ker\,\sigma_T \arrow[r, two heads] \arrow[rrd, "f^*\gamma"'] & ker\,f^*\sigma_T \arrow[r, hook] & \mathcal{E}'_P \arrow[r, two heads, "f^*\sigma_T"] \arrow[d, "f^*q"] & \mathcal{E}_P \arrow[d] \arrow[ld, dashed] \\ & & f^*\mathcal{F} \arrow[r, two heads] & coker\,f^*\gamma \end{tikzcd} \qquad \begin{tikzcd} ker\,\sigma_T \arrow[r, hook] \arrow[rd, "\gamma"'] & \mathcal{E}'_T \arrow[r, two heads, "\sigma_T"] \arrow[d, "q"] & \mathcal{E}_T \arrow[d] \arrow[ld, dashed] \\ & \mathcal{F} \arrow[r, two heads] & coker\,\gamma \end{tikzcd}$$

f^*q factors through $f^*\sigma_T$, i.e. $f^*\langle \mathcal{F}, q \rangle \in \mathfrak{Grass}_S(\mathcal{E}, d)(P)$, if and only if its restriction to the kernel $ker\,f^*\sigma_T$ is zero. By precomposing with the surjection from $f^*ker\,\sigma_T$, which is a result of the right exactness of the pullback, this holds if and only if $f^*ker\,\sigma_T \to f^*\mathcal{F}$ is zero, which is to say $f^*\mathcal{F} \cong coker\,f^*\mathcal{F}$. This map is precisely the pullback $f^*\gamma$ of the composition $\gamma : ker\,\sigma_T \hookrightarrow \mathcal{E}'_T \twoheadrightarrow \mathcal{F}$ on T.

Therefore, we are searching for a closed subscheme C such that for every morphism of S-schemes $f : P \to T$, f factors through C if and only if $f^*\gamma$ is zero. We prove the existence of such a subscheme in the appendix A.2. $\square$

Corollary 2.8 *For any finite collection $\{\mathcal{E}_i\}_i$ of coherent sheaves on S, the natural projections from their direct sum induce the closed embedding of the coproduct*

$$\coprod_i \mathfrak{Grass}_S(\mathcal{E}_i, d) \hookrightarrow \mathfrak{Grass}_S(\textstyle\bigoplus_i \mathcal{E}_i, d)\,.$$

Theorem 2.9 *The functor $\mathfrak{G}rass_S(\mathcal{E}, d)$ is representable by a proper scheme $Grass_S(\mathcal{E}, d)$ over S.*

Proof We follow the approach sketched in [Nit05, 1.(2)]. For the reduction step we can use again the representability criterion 1.7 since $\mathfrak{G}rass_S(\mathcal{E}, d)$ is a Zariski sheaf, as in the proof of 2.2.

Since $\mathcal{E}$ is coherent, it is globally generated on small enough open subsets. If we restrict attention to small enough S-schemes T, i.e. those whose structure morphism factors through an open subset where $\mathcal{O}_S^{\oplus r} \twoheadrightarrow \mathcal{E}$ for some r, the functor would be representable because it would coincide with a closed subscheme of $\mathfrak{G}rass_S(\mathcal{O}_S^{\oplus r}, d)$, in light of Lemma 2.7. So we fix an open cover $\{U_i\}_i$ of S of such open subsets for $\mathcal{E}$.

For each U_i define the functor $\mathfrak{G}rass_S(\mathcal{E}, d)_i$ that associates to each S-scheme T the same set $\mathfrak{G}rass_S(\mathcal{E}, d)(T)$ if T factors through U_i, and the empty set if it doesn't. It's just a matter of unwrapping definitions to prove that these are a cover of open subfunctors of $\mathfrak{G}rass_S(\mathcal{E}, d)$. This becomes apparent after noticing that $\mathfrak{G}rass_S(\mathcal{E}, d)_i = \mathfrak{G}rass_S(\mathcal{E}, d) \times_{h_S} h_{U_i}$ in the contravariant functor category of S-schemes, in which h_S is the final object and h_{U_i} is empty precisely on S-schemes that don't factor through U_i.

Take now an arbitrary i. Since on U_i we have a surjection $\mathcal{O}_{U_i}^{\oplus r} \twoheadrightarrow \mathcal{E}|_{U_i}$ for some r, with 2.7 we get the following maps of sets natural in T:

$$\mathfrak{G}rass_S(\mathcal{E}, d)_i(T) \cong \mathfrak{G}rass_{U_i}(\mathcal{E}|_{U_i}, d)(T) \hookrightarrow \mathfrak{G}rass_{U_i}(\mathcal{O}_{U_i}^{\oplus r}, d)(T) \cong \mathfrak{G}rass_{U_i}(r, d)$$

Thus, we can represent $\mathfrak{G}rass_S(\mathcal{E}, d)_i$ by a closed subscheme of the U_i-scheme $Grass_{U_i}(r, d)$ considered as an S-scheme through the inclusion $U_i \hookrightarrow S$. Indeed, for any S-scheme T, if T factors through U_i then the morphisms of S-schemes $f : T \to Grass_{U_i}(r, d)$ are precisely the morphisms of U_i-schemes; on the other hand, if T doesn't factor through U_i then there actually isn't any such morphism of S-schemes.

$$\begin{tikzcd} \mathfrak{G}rass_S(\mathcal{E}, d)_i \arrow[r] \arrow[d] & \mathfrak{G}rass_S(\mathcal{E}, d) \arrow[d] \\ h_{U_i} \arrow[r] & h_S \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "f"] \arrow[dr, dashed] \arrow[drr] & Grass_{U_i}(r, d) \arrow[d] \arrow[dr] & \\ & U_i \arrow[r, "\iota"] & S \end{tikzcd}$$

So each of the open subfunctors $\mathfrak{G}rass_S(\mathcal{E}, d)_i$ is representable by a closed subscheme of the S-scheme $Grass_{U_i}(r, d)$, for some r depending on i. Thus, by 1.7, $\mathfrak{G}rass_S(\mathcal{E}, d)$ is representable by an S-scheme $Grass_S(\mathcal{E}, d)$.

Since $Grass(r, d)$ is proper over $\mathbb{Z}$ and proper morphisms are stable under base extension, with 2.6 follows that all the schemes $Grass_S(r, d)$ are proper over S. As properness is a local property on the base, the general scheme $Grass_S(\mathcal{E}, d)$ is proper over S because it is obtained as a gluing of closed subschemes of the schemes $Grass_{U_i}(r, d)$ along open covers of S. $\square$

We will denote by $\pi : Grass_S(\mathcal{E}, d) \to S$ the proper structure morphism and $\pi^*\mathcal{E} \twoheadrightarrow \mathcal{Q}$ the tautological quotient bundle on $Grass_S(\mathcal{E}, d)$.

2.4 Projective schemes

In the literature there are notions of projectivity weaker in general than the one used in Hartshorne [Har77, II.4]. There, a morphism $X \to S$ is called projective if it factors through a closed embedding into some projective space $\mathbb{P}^r_S$ over S, which is equivalent to the notion of $H - projectivity$ below.

Fix again a base scheme S and a coherent sheaf $\mathcal{E}$ on it. The *projective bundle* $\mathbb{P}_S(\mathcal{E})$ *of* $\mathcal{E}$ *over* S is constructed in [Har77, II.7] as $\mathbf{Proj}_S(Sym_S(\mathcal{E}))$, where $Sym_S(\mathcal{E})$ denotes the symmetric graded algebra of $\mathcal{E}$. After the explicit construction is carried out, Hartshorne proves in [Har77, II.7.12] that the functor of points of this scheme is naturally isomorphic to our relative Grassmannian functor $\mathfrak{G}rass_S(\mathcal{E}, 1)$ defined above. For the purposes of this paper, we don't need to get into the details of the *relative projective space* $\mathbf{Proj}_S\mathcal{I}$ of a graded $\mathcal{O}_S$-algebra $\mathcal{I}$. So we define $\mathbb{P}_S(\mathcal{E})$ directly as $Grass_S(\mathcal{E}, 1)$, since they are uniquely isomorphic as S-schemes by Yoneda's lemma. The tautological bundle on $\mathbb{P}_S(\mathcal{E})$ is then denoted by $\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$ and more commonly called the twisting sheaf.

Definition 2.10 *An S-scheme T is called:*

- projective, *if there exists a closed embedding of T into $\mathbb{P}_S(\mathcal{E})$ over S, for some coherent sheaf $\mathcal{E}$ on S,*
- strongly projective, *if the above holds for some vector bundle $\mathcal{E}$ on S,*
- H-projective, *if the above holds for some finite rank free sheaf $\mathcal{O}_S^{\oplus r}$ on S.*

These properties do not in general coincide. As in the proof of Theorem 2.9, on trivializing open subsets $U \subseteq S$ for $\mathcal{E}$ we do have closed embeddings $\mathbb{P}_U(\mathcal{E}|_U) \hookrightarrow \mathbb{P}_U(\mathcal{O}_U^{\oplus r})$; but this doesn't imply that they somehow glue to an embedding of the whole $\mathbb{P}_S(\mathcal{E})$ into a single bigger projective space $\mathbb{P}^r_S$.

Definition 2.11 *We call a line bundle $\mathcal{L}$ on an S-scheme T* relatively very ample *if there exists a coherent sheaf $\mathcal{E}$ on S and a locally closed immersion $j : T \hookrightarrow \mathbb{P}_S(\mathcal{E})$ of S-schemes so that $\mathcal{L} \cong j^*\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$.*

Lemma 2.12 *Let S be a noetherian scheme. A proper S-scheme T is projective if and only if it has a relatively very ample line bundle.*

Proof Since T and $\mathbb{P}_S(\mathcal{E})$ are proper over S, every immersion of the first into the second is a closed embedding by [Sta24, 04XV] and [Sta24, 01W6]. In addition, a line bundle on T gives the desired coherent sheaf on S through the direct image, by [Har77, II.5.8]. □

$$\begin{tikzcd}
 & \mathcal{O}_{\mathbb{P}_S(\tau^*\mathcal{L})}(1) \arrow[d, squiggly, dash] \\
\mathcal{L} \arrow[d, squiggly, dash] & \mathbb{P}_S(\tau_*\mathcal{L}) \arrow[d, "\pi"] \\
T \arrow[ur, hook, "\text{closed}"] \arrow[r, "\text{proper}", "\tau"'] & S
\end{tikzcd}$$

Proposition 2.13 *$Grass_S(\mathcal{E}, d)$ is projective over S. If $\mathcal{E}$ is a vector bundle (resp. free), $Grass_S(\mathcal{E}, d)$ is strongly projective (resp. H-projective) over S.*

Proof Note that $\Lambda^d\mathcal{E}$ is also coherent, and it is a vector bundle (resp. free) if $\mathcal{E}$ is. Then, as in 2.4, we can express $\mathfrak{G}rass_S(\mathcal{E}, d)$ as a subfunctor of $\mathfrak{G}rass_S(\Lambda^d\mathcal{E}, 1)$. The corresponding natural transformation between the functors of points will then yield a monomorphism of schemes, which by the properness verified above will be a closed embedding.

$$\mathcal{P} : Grass_S(\mathcal{E}, d) \to Grass_S(\Lambda^d\mathcal{E}, 1)$$

The natural transformation $\mathcal{N}$ in 2.5 works for the relative Grassmannian without change. We just have to check again that it expresses a subfunctor. And again, the restrictions of $\mathcal{N}$ to the open subfunctors of $\mathfrak{G}rass_S(\mathcal{E}, d)$ considered in the proof of 2.9 map consistently into the open subfunctors of $\mathfrak{G}rass_S(\Lambda^d\mathcal{E}, 1)$, as trivializing open covers of S for $\mathcal{E}$ work also for $\Lambda^d\mathcal{E}$, meaning we get the same decomposition in open subfunctors. Therefore, it will be enough to check that the individual restrictions express subfunctors. Indeed, if $\mathcal{E}$ is a quotient of $\mathcal{O}_S^{\oplus r}$ on U_i, the restriction $\mathcal{N}_{U_i}$ on $\mathfrak{G}rass_S(\mathcal{E}, d)_i$ fits into the diagram of subfunctors

$$\begin{tikzcd}
\mathfrak{G}rass_S(\mathcal{E}, d)_i \arrow[r, "\mathcal{N}"] \arrow[d, hook, "\text{closed}"] & \mathfrak{G}rass_S(\Lambda^d\mathcal{E}, 1)_i \arrow[d, hook, "\text{closed}"] \\
\mathfrak{G}rass_S(\mathcal{O}_S^{\oplus r}, d)_i \arrow[r, hook] & \mathfrak{G}rass_S(\Lambda^d\mathcal{O}_S^{\oplus r}, d)_i
\end{tikzcd}$$

Consequently, $\mathcal{N}_{U_i}$ itself expresses a subfunctor for all U_i. □

Chapter 3

Flattening stratification

This chapter is a technical interlude in our paper, aimed at recalling many important facts about Hilbert polynomials and flat morphisms which explain why we are interested in flat families of subschemes. As we have seen in 1.15, flat families are related to coherent sheaves $\mathcal{F}$ on a scheme X which are flat over the base scheme T, the parameter space of the family. Thus we consider for the rest of this chapter the following objects, illustrated in the diagram below. Let T be a locally noetherian scheme, $\langle X, \pi \rangle$ a projective T-scheme and $\mathcal{F}$ a coherent sheaf on X. Fix a relatively very ample line bundle $\mathcal{L}$ on X.

$$
\begin{tikzcd}
 & \mathcal{F}|_t \arrow[r, squiggly, -|] & X_t \arrow[r, hook] \arrow[d] & X \arrow[r, squiggly, |-] \arrow[d, "\pi"] & \mathcal{F} \\
(\pi_*\mathcal{F})|_t \arrow[r] & \Gamma(X_t, \mathcal{F}|_t) \arrow[r, squiggly, -|] & Spec\, k(t) \arrow[r, hook] & T \arrow[r, squiggly, |-] & \pi_*\mathcal{F}
\end{tikzcd}
$$

3.1 Flat coherent sheaves

The Hilbert polynomial of a coherent sheaf on a projective scheme over a field encodes important information about its schematic support, most notably its dimension, degree and arithmetic genus. In our setting, for each parameter $t \in T$, we have the coherent sheaf $\mathcal{F}|_t$ on X_t, which is projective over $k(t)$ with very ample line bundle $\mathcal{L}|_t$. In this context, for every $z \in \mathbb{Z}$, we will denote by $\mathcal{F}(z)$ and $\mathcal{F}|_t(z)$ the tensored sheaves $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes z}$ and $\mathcal{F}|_t \otimes_{\mathcal{O}_{X_t}} \mathcal{L}|_t^{\otimes z}$ respectively.

Definition 3.1 *The Hilbert polynomial $\mathcal{H}^{\mathcal{L}}_{\mathcal{F},t}$ of $\mathcal{F}$ at $t \in T$, with respect to the line bundle $\mathcal{L}$, is the numerical polynomial*

$$z \in \mathbb{Z} \mapsto \sum_{i=0}^{\infty}(-1)^i dim_{k(t)} H^i(X_t, \mathcal{F}|_t(z)) \ \in \mathbb{Z}\,.$$

The definition of $\mathcal{H}^{\mathcal{L}}_{\mathcal{F},t}$ is well-founded because the numerical function above indeed defines an element of the polynomial ring $\mathbb{Q}[z]$ for all $t \in T$ and very ample line bundles $\mathcal{L}|_t$ on X_t. This can be proved by induction on the dimension of $Supp\, \mathcal{F}|_t$ in $X|_t$ as follows. For each $z \in \mathbb{Z}$, $\mathcal{H}^{\mathcal{L}}_{\mathcal{F},t}(z)$ is the Euler characteristic $\chi(\mathcal{F}|_t(z))$, which has an additivity property along exact sequences of sheaves. Then, by finding suitable sequences containing $\mathcal{F}$ and sheaves of lower dimensional support, we can use the induction assumption to relate the quantity $\mathcal{H}^{\mathcal{L}}_{\mathcal{F},t}(z) - \mathcal{H}^{\mathcal{L}}_{\mathcal{F},t}(z-1)$ to a numerical polynomial. From a combinatorial argument it follows that $\mathcal{H}^{\mathcal{L}}_{\mathcal{F},t}(z)$ itself is a numerical polynomial. For the details we refer to [Har77, III.5,Ex2].

We can consider the Hilbert polynomial of $\mathcal{F}$ as a function of $t \in T$, which for each parameter gives us information about the corresponding subscheme of X_t. The following theorem then shows why flatness of $\mathcal{F}$ over T is a desirable property for a family of subschemes: the information about these subschemes determined by the Hilbert polynomial is locally indipendent of the parameter. In particular, if T is connected, this information is completely independent.

Theorem 3.2 *If $\mathcal{F}$ is flat over T, then the function $t \in T \mapsto \mathcal{H}^{\mathcal{L}}_{\mathcal{F},t} \in \mathbb{Q}[z]$ is locally constant. If T is reduced, the converse also holds.*

Proof This is proved in [Har77, III.9.9], one just has to reduce to the case where T is an irreducible noetherian scheme, and note that in Hartshorne's proof the first implication follows also without the reducedness assumption on T. $\square$

In the particular case $X = T$, which is in fact a projective T-scheme with very ample line bundle $\mathcal{O}_T$, coherent sheaves on X which are flat over T are precisely the finite rank locally free sheaves. This is the consequence of a commutative algebra fact explained in [Har77, III.9.1A]. In this case, the Hilbert polynomial at $t \in T$ of a coherent flat sheaf on T is precisely its free rank in a neighbourhood of t, which in fact is obviously locally constant.

The following lemma gives us a way of relating the flatness of sheaves on an arbitrary projective scheme X to the flatness, i.e. local freeness, of their projections onto T. This is the crucial result used for the reduction to the case $X = T$ in the proof of the flattening stratification theorem and of the the main existence theorem in Chapter 4.

Lemma 3.3 *If there is an $N \in \mathbb{Z}$ is such that $R^i\pi_*\mathcal{F}(z) = 0$ for all $i \geq 1, z \geq N$, then $\mathcal{F}$ is flat over T if and only if $\pi_*\mathcal{F}(z)$ is locally free for all $z \geq N$.*

Proof The assertion is local on T, as for any open subscheme $U \subseteq T$ we can focus on the coherent sheaf $(\pi_*\mathcal{F}(z))|_U = \pi_*(\mathcal{F}|_{X_U}(z))$ on $X_U = X \times_T U$ over U.

Having assumed that X is projective over T, we can take a closed embedding $X \hookrightarrow \mathbb{P}_T(\mathcal{V})$ of T-schemes which pulls back the twisting sheaf $\mathcal{O}(1)$ to $\mathcal{L}$, for some coherent sheaf $\mathcal{V}$ on T. So by restricting to a suitable open cover of T, we can assume without loss of generality that $\mathcal{V}$ is globally generated. Then, any surjection $\mathcal{O}_T^{\oplus n+1} \twoheadrightarrow \mathcal{V}$ yields by 2.7 a closed embedding $\mathbb{P}_T(\mathcal{V}) \hookrightarrow \mathbb{P}^n_T$; denote by $\iota : X \hookrightarrow \mathbb{P}^n_T$ its composition with the embedding above.

Cohomology and higher direct images on the closed subscheme X can be computed on the ambient scheme $\mathbb{P}^n_T$ through the direct image ι_* [Har77, III.2.10]. Since $\mathcal{F}$ is flat over T if and only if the direct image $\iota_*\mathcal{F}$ on $\mathbb{P}^n_T$ is flat over T, and their projection $\pi_*\mathcal{F}$ onto T is the same by commutativity, the assertion reduces to the H-projective case $X = \mathbb{P}^n_T$.

We can restrict further to the irreducible components of T, thus assume that $T = Spec\, A$ for a local noetherian ring A. The claim restricts accordingly to: $\mathcal{F}$ is flat over T if and only if $\Gamma(\mathbb{P}^n_T, \mathcal{F}(z))$ is a free A-module for all $z \geq N$. This is a step in the proof of [Har77, III.9.9], which we already mentioned for 3.2 above. Here, N is the lower bound for Serre vanishing, which is guaranteed to exist by [Har77, III.8.8] if T is noetherian. □

Theorem 3.4 *If $X = \mathbb{P}^n_T$ and T is noetherian and integral, then there exists a non-empty open subscheme $U \subseteq T$ such that $\mathcal{F}|_{\mathbb{P}^n_U}$ is flat over U.*

Proof After passing to a non-empty open affine subset of T, the theorem is a consequence of the following commutative algebra fact.
Let A be a noetherian integral domain, B a finite type A-algebra and M a finite type B-module. Then there exists an $f \in A - 0$, such that the localisation M_f is a free A_f-module. For the proof of this we refer to [Nit05, 4.1]. □

3.2 Flattening stratification theorem

In this section we will denote by $\mathcal{H}_\mathcal{F}$ the subset of the numerical polynomials in $\mathbb{Q}[z]$ which occur as the Hilbert polynomial $\mathcal{H}_{\mathcal{F},t}$ for some $t \in T$. For any morphism $f : P \to T$, we denote by $f_X : X_P \to X$ its base extension by $\pi : X \to T$, so that $f_X^*\mathcal{F}$ will denote the pullback on X_P.

Theorem 3.5 *a) If T is noetherian, the subset $\mathcal{H}_\mathcal{F}$ is finite.*

b) For each $\Phi \in \mathbb{Q}[z]$, there exists a locally closed subscheme $\iota_\Phi : T_\Phi \hookrightarrow T$ satisfying the following universal property:

For every morphism $f : P \to T$, f factors through ι_Φ if and only if the coherent sheaf $f_X^*\mathcal{F}$ on X_P is flat over P with Hilbert polynomial $\mathcal{H}_{f_X^*\mathcal{F},p} = \Phi$ for all $p \in P$.

In particular, each set T_Φ consists precisely of the points $t \in T$ where $\mathcal{H}_{\mathcal{F},t} = \Phi$, so their disjoint union is a set-theoretic partition of T and $\Phi \in \mathcal{H}_\mathcal{F}$ if and only if T_Φ is non empty.

c) The scheme-theoretic coproduct $\widetilde{T}$ of the schemes T_Φ, which comes equipped with the glued morphism $\widetilde{\iota} : \widetilde{T} \to T$, satisfies the following universal property:

For every morphism $f : P \to T$, f factors through $\widetilde{\iota}$ if and only if the coherent sheaf $f_X^*\mathcal{F}$ on X_P is flat over P.

$$\widetilde{\iota} \colon\ \widetilde{T} = \coprod_{\Phi \in \mathcal{H}_\mathcal{F}} T_\Phi \longrightarrow T$$

We call the subschemes T_Φ the *strata* and $\widetilde{T}$ the *flattening stratification* of T with respect to the sheaf $\mathcal{F}$ on X. For the proof we follow our primary reference [Nit05, 4.3].

Proof **Equivalence between claims** $b)$ **and** $c)$

Assuming the existence of the subschemes T_Φ we can prove the universal property of $\widetilde{T}$. Any morphism $f : P \to T$ that factors through $\widetilde{\iota}$ also breaks up P as the coproduct $\coprod_\Phi P_\Phi$ of preimages of each T_Φ, implying by claim $b)$ that $f_X^*\mathcal{F}$ is flat on each component P_Φ. Conversely, if $f_X^*\mathcal{F}$ is flat over P then $\mathcal{H}_{f_X^*\mathcal{F},p}$ is constant on the connected components of P; so these components can be grouped in subschemes P_Φ according to the Hilbert polynomial, and each of them factors through T_Φ under f by the universal property.

Assuming the existence of a scheme $\widetilde{T}$ with the universal property in claim $c)$, the subschemes T_Φ can be recovered as a grouping of the connected components of $\widetilde{T}$ according to the Hilbert polynomial of $\widetilde{\iota}_X^*\mathcal{F}$, which is a coherent sheaf flat over $\widetilde{T}$.

The claim about the set-theoretic nature of each T_Φ follows by applying its universal property to the residue fields $Spec\, k(t) \hookrightarrow T$.

Reduction to the noetherian H-projective case $X = \mathbb{P}^n_T$

Claim $a)$ is local on T as for any open cover $\{U_i\}_i$ of T, if T is noetherian the set $\mathcal{H}_{\mathcal{F}} = \bigcup_i \mathcal{H}_{\mathcal{F}|U_i}$ is finite. Since $f_X^*\mathcal{F}$ is flat over P if and only if its restrictions to an open cover of P are flat over P, the functor corresponding to the universal property in claim $c)$ is a Zariski sheaf. Therefore, by 1.8 it is enough to find $\widetilde{T}$ on an open cover of T.

So we can assume without loss of generality that T is noetherian and there is a closed embedding $\iota : X \hookrightarrow \mathbb{P}^n_T$, as in the proof of 3.3.

For every morphism $f : P \to T$, the sheaf $f_X^*\mathcal{F}$ on X_P is flat over P if and only if the direct image $\iota_{P*}f_X^*\mathcal{F} = f_{\mathbb{P}}^*\iota_*\mathcal{F}$ on $\mathbb{P}^n_P$ is flat over P, so the flattening stratification we are looking for is precisely the one with respect to the coherent sheaf $\iota_*\mathcal{F}$ on $\mathbb{P}^n_T$. Therefore, it is enough to find $\widetilde{T}$ in the H-projective case $X = \mathbb{P}^n_T$ with $\mathcal{L} = \mathcal{O}(1)$. Claim $a)$ also reduces because the Hilbert polynomials of $\iota_*\mathcal{F}$ and $\mathcal{F}$ are the same.

The case $n = 0$

The theorem is proved by reducing to the case $n = 0$. So we start by assuming $X = T$. As the theorem is local on T, we can restrict without loss of generality to neighbourhoods of an arbitrary point $t \in T$.

By Nakayama's lemma applied to the coherent sheaf $\mathcal{F}$ on a small enough neighbourhood of t, we can assume that $\mathcal{F}$ fits in the following exact sequence, where $r = dim_{k(t)}\mathcal{F}|_t$.

$$\mathcal{O}_P^{\oplus m} \xrightarrow{f^*\gamma} \mathcal{O}_P^{\oplus r} \xrightarrow{f^*q} f^*\mathcal{F} \to 0 \qquad \mathcal{O}_T^{\oplus m} \xrightarrow{\gamma} \mathcal{O}_T^{\oplus r} \xrightarrow{q} \mathcal{F} \to 0$$

For every morphism $f : P \to T$, $f^*\mathcal{F}$ is flat over P if and only if it is locally free. Since $f^*\mathcal{F}$ fits in the exact sequence above, it is locally free of rank r if and only if $f^*\gamma$ is the zero map, so if and only if f factors through a closed subscheme $\widetilde{T}$, whose existence is guaranteed by A.2 applied to the map γ.

Claim $a)$ and properties of $\pi_*\mathcal{F}$

Take a non-empty irreducible open subset $U \subseteq T$, for example by removing from T all but one of the finitely many irreducible components. By giving U the reduced subscheme structure, we can apply 3.4 to obtain a non-empty open subscheme $V_1 \subseteq T$ such that $\mathcal{F}|\mathbb{P}^n_{V_1}$ is flat over V_1. We can repeat this argument for the closed subscheme $T - V_1 \subseteq T$ with its reduced structure, to obtain through noetherian induction a finite sequence of reduced, locally closed subschemes $\{V_j\}_j$ of T, which compose a set-theoretic partition of T and such that each $\mathcal{F}_j = \mathcal{F}|\mathbb{P}^n_{V_j}$ is flat over V_j.

Each residue field $Spec\, k(t) \hookrightarrow T$ factors through the locally closed subscheme V_j with $t \in V_j$. So computing the cohomologies of the fibers reduces to each individual V_j: $\mathcal{F}|_t = \mathcal{F}_j|_t$, and in particular we have $\mathcal{H}_{\mathcal{F},t} = \mathcal{H}_{\mathcal{F}_j,t}$ for all $t \in V_j$. Knowing this, we can conclude three useful facts about $\mathcal{F}$ over T by using its flatness properties when restricted to the subschemes V_j.

$$\begin{tikzcd}
\mathcal{F}|_t \arrow[d, squiggly] & \mathcal{F}_j \arrow[d, squiggly] & \mathcal{F} \arrow[d, squiggly] \\
\mathbb{P}^n_{k(t)} \arrow[r, hook] \arrow[d] & \mathbb{P}^n_{V_j} \arrow[r, hook] \arrow[d, "\pi_j"] & \mathbb{P}^n_T \arrow[d, "\pi"] \\
Spec\, k(t) \arrow[r, hook] & V_j \arrow[r, hook] & T
\end{tikzcd}$$

(A) Since the flat sheaves $\mathcal{F}_j$ have constant Hilbert polynomial on the finitely many connected components of the finitely many V_j, $\mathcal{H}_\mathcal{F}$ must be finite. So claim $a)$ is verified.

Having assumed that T is noetherian, we can take a lower bound $N \in \mathbb{N}$ for Serre vanishing [Har77, III.8.8].

B) For all $i \geq 1$, $z \geq N$ we have $R^i\pi_*\mathcal{F}(z) = 0$, and in particular $H^i(\mathbb{P}^n_{k(t)}, \mathcal{F}|_t(z)) = 0$ for all $t \in T$.

Consequently, also for each j and $t \in V_j$ it holds $H^i(\mathbb{P}^n_{k(t)}, \mathcal{F}_j|_t(z)) = 0$. With the semicontinuity theorems [Har77, III.12.9-11] applied to each of the flat sheaves $\mathcal{F}_j$ over V_j we get the following.

$$\forall i \geq 1, z \geq N, t \in V_j : \quad R^i\pi_{j*}\mathcal{F}_j(z) = 0 \quad (\pi_{j*}\mathcal{F}_j(z))|_t \xrightarrow{\sim} \Gamma(\mathbb{P}^n_{k(t)}, \mathcal{F}_j|_t(z))$$

Since the locally closed subschemes V_j are not in general flat over T, we can't apply the usual flat base change properties to compute the fibers $(\pi_*\mathcal{F})|_t$ using its restrictions $\mathcal{F}_j$. But for a big enough tensor product with the twisting sheaf, depending upon the subscheme V_j, the base change relations hold nonetheless. We defer this result to A.4. So take the $N \in \mathbb{N}$ big enough, common to the finitely many V_j, so that we have $(\pi_*\mathcal{F}(z))|_{V_j} \xrightarrow{\sim} \pi_{j*}\mathcal{F}_j(z)$ for all $z \geq N$. In particular, we have $(\pi_*\mathcal{F}(z))|_t \cong (\pi_{j*}\mathcal{F}_j(z))|_t$, which combined with the identities above yields the following property of $\pi_*\mathcal{F}$.

(C) For all $t \in T$ and $z \geq N$ we have a natural isomorphism of $\mathcal{H}_{\mathcal{F},t}(z)$-dimensional $k(t)$-vector spaces $(\pi_*\mathcal{F}(z))|_t \xrightarrow{\sim} \Gamma(\mathbb{P}^n_{k(t)}, \mathcal{F}|_t(z))$.

Flatness of $f_X^*\mathcal{F}$ and local freeness of $f^*\pi_*\mathcal{F}$

Let $f : P \to T$ be a morphism of locally noetherian schemes. By Lemma 3.3, $f_X^*\mathcal{F}$ is flat over P if and only if $\pi_{P*}f_X^*\mathcal{F}(z)$ is locally free for all $z >> 0$. Since the universal property in claim $c)$ of the theorem has to hold for an arbitrary, not necessarily flat f, we must use the base-change without flatness A.4 again: the natural homomorphism $f^*\pi_*\mathcal{F}(z) \to \pi_{P*}f_X^*\mathcal{F}(z)$ is an isomorphism for all $z >> 0$, with lower bound depending on f.

Unfortunately, this is not quite enough because in the reduction step below, we will need to have a lower bound for z forcing this condition indipendently of f. Nonetheless, we claim that such a bound can be chosen as the N found in the preceding step, which indeed depends only on $\mathcal{F}$. To be precise, we claim the following equivalence:

For every morphism $f : P \to T$, $f_X^*\mathcal{F}$ is flat over P if and only if the coherent sheaf $f^*\pi_*\mathcal{F}(z)$ is locally free for all $z \geq N$.

For one direction A.4 is enough: if $f^*\pi_*\mathcal{F}(z)$ is locally free for all $z \geq N$ then so is $\pi_{P*}f_X^*\mathcal{F}(z)$ for all $z >> 0$, and thus $f_X^*\mathcal{F}$ is flat.

Assume now that $f_X^*\mathcal{F}$ is flat over P. We claim that given such a morphism f and an N satisfying properties (B) and (C) above, N works as a lower bound for Serre vanishing of $f_X^*\mathcal{F}$ and the base-change $f^*\pi_*\mathcal{F}(z) \to \pi_{P*}f_X^*\mathcal{F}(z)$ is an isomorphism for all $z \geq N$. Therefore, also $f^*\pi_*\mathcal{F}(z)$ is locally free for all $z \geq N$.

Since it is due to the properties (B) and (C) of $\pi_*\mathcal{F}$ that we have the converse implication, it would make sense to prove the claim in this section. However, we defer the details to the appendix A.6 because this criterion will be employed again later in the proof of the main existence theorem 4.14.

Reduction to the case $n = 0$

Let now n be arbitrary, and take the N above big enough for the properties of the direct image $\pi_*\mathcal{F}$ proved above to hold. By the preceding step, to prove claim $c)$ of the theorem we have to find a locally closed subscheme $\widetilde{T} \hookrightarrow T$ with the following universal property:

For every morphism $f : P \to T$, f factors through $\widetilde{T}$ if and only if the coherent sheaf $f^*\pi_*\mathcal{F}(z)$ is locally free on P for all $z \geq N$.

We reduce to the case $n = 0$ by noting that such a $\widetilde{T}$ is a kind of intersection of the flattening stratifications of T with respect to all the sheaves $\pi_*\mathcal{F}(z)$. So we consider the sequence of coherent sheaves $\mathcal{E}_j = \pi_*\mathcal{F}(N+j)$ on T, for $j \geq 0$. Take the flattening stratification $\widetilde{T_0}$ of T with respect to the sheaf $\mathcal{E}_0$ and the flattening stratification $\widetilde{T_j}$ of $\widetilde{T_{j-1}}$ with respect to $\mathcal{E}_j|_{\widetilde{T_{j-1}}}$, iteratively for all $j \geq 1$. As we will see, this sequence of nested subschemes becomes eventually stationary and its limit satisfies the desired universal property.

First, we prove that for all $j \geq 0$, the Hilbert polynomial of $\mathcal{E}_j|_{\widetilde{T_n}}$ over $\widetilde{T_n}$, i.e. its fiber dimension, is a locally constant function of $t \in \widetilde{T_n}$. Recall that since we are in the case $n = 0$ the strata of each $\widetilde{T_j}$ are indexed by natural numbers, since the Hilbert polynomials correspond to the rank as a locally free sheaf of the flat sheaves on $\widetilde{T_j}$.
Given $t \in \widetilde{T_n}$, we can take note for each $0 \leq j \leq n$ of which stratum of $\widetilde{T_j}$ contains t, yielding a sequence of $n+1$ numbers $(e_j)_{0 \leq j \leq n}$. Take the following intersection of open neighbourhoods of t in $\widetilde{T_n}$: $U = \bigcap_{0 \leq j \leq n} \iota_j^{-1}(\widetilde{T_j}_{e_j})$, where $\iota_j : \widetilde{T_n} \hookrightarrow \widetilde{T_j}$ is the inclusion and $\widetilde{T_j}_{e_j}$ is the stratum e_j of $\widetilde{T_j}$.

$$\begin{tikzcd}
\mathcal{E}_0|_U, \mathcal{E}_1|_U, \ldots \arrow[d, rightsquigarrow] & \widetilde{T_n} \arrow[r, hook] & \cdots \arrow[r, hook] & \widetilde{T_1} \arrow[r, hook] & \widetilde{T_0} \arrow[dr] & \mathcal{E}_0, \mathcal{E}_1, \ldots \arrow[d, rightsquigarrow] \\
U \arrow[r, equal] & \bigcap\limits_{j=0}^{n} \widetilde{T}_{j_{e_j}} \arrow[u, hook] \arrow[r, hook] & \cdots \arrow[r, hook] & \bigcap\limits_{j=0}^{1} \widetilde{T}_{j_{e_j}} \arrow[u, hook] \arrow[r, hook] & \widetilde{T}_{0e_0} \arrow[u, hook] \arrow[r, hook] & T
\end{tikzcd}$$

Then, by the universal property of each stratum $\widetilde{T}_{j_{e_j}}$, we have that $\mathcal{E}_j|_U$ is locally free of rank e_j everywhere over U, so $e_j = dim_{k(t)}\mathcal{E}_j|_t$ for all $0 \leq j \leq n$. By properties (B) and (C) of $\pi_*\mathcal{F}(z)$ we have the following equalities.

$$\forall t \in U, 0 \leq j : \quad dim_{k(t)}\mathcal{E}_j|_t \overset{(C)}{=} dim_{k(t)}\Gamma(\mathbb{P}^n_{k(t)}, \mathcal{F}|_t(N+j)) \overset{(B)}{=} \mathcal{H}_{\mathcal{F},t}(N+j)$$

So our Hilbert polynomial $\mathcal{H}_{\mathcal{F},t}$ has fixed values $(e_j)_j$ at the $n+1$ arguments $(N+j)_j$. It is a famous combinatorial fact that a numerical polynomial of degree $\leq n$ is uniquely determined by its integer values at $n+1$ distinct points, so $\mathcal{H}_{\mathcal{F},t}$ is a constant function of $t \in U$.
Thus, $\mathcal{E}_j|_{\widetilde{T_n}}$ has constant Hilbert polynomial as a function of $t \in U$; since t was arbitrary and U is open, it is locally constant on $\widetilde{T_n}$.

Now we can prove that for all $j \geq n$, $\widetilde{T}_j$ is actually a closed subscheme of $\widetilde{T_n}$. It is a consequence of the fact that the reduced subscheme structure $\widetilde{T_n}_{red}$, which is a closed subscheme of $\widetilde{T_n}$, factors through each $\widetilde{T}_j$. Indeed, this follows by the universal property of each $\widetilde{T}_j$, as the restriction of $\mathcal{E}_j|_{\widetilde{T_n}}$ on the reduced structure has locally constant Hilbert polynomial, so it is flat by the converse implication in theorem 3.2.

$$\begin{tikzcd}
\mathcal{E}_0|_{\widetilde{T}}, \mathcal{E}_1|_{\widetilde{T}}, \ldots \arrow[d, squiggly] & \widetilde{T_n}_{red} \arrow[d] \arrow[dr] \arrow[drr, hook] & & \mathcal{E}_0|_{\widetilde{T_n}}, \mathcal{E}_1|_{\widetilde{T_n}}, \ldots \arrow[d, squiggly] & & \mathcal{E}_0, \mathcal{E}_1, \ldots \arrow[d, squiggly] \\
\widetilde{T} \arrow[r, hook, "\text{closed}"'] & \cdots \arrow[r, hook, "\text{closed}"'] & \widetilde{T_{n+1}} \arrow[r, hook, "\text{closed}"'] & \widetilde{T_n} \arrow[r, hook, "\text{locally closed}"'] & \cdots \arrow[r, hook] & T
\end{tikzcd}$$

$$\mathcal{I} \hookleftarrow \cdots \hookleftarrow \mathcal{I}_{n+1} \hookleftarrow \mathcal{I}_n = 0 \hookrightarrow \mathcal{O}_{\widetilde{T_n}}$$

So the sequence of nested subschemes $(\widetilde{T}_j)_{j \geq 0}$ is actually one of nested closed subschemes after the n^{th} term. Since $\widetilde{T_n}$ is noetherian, such a sequence must become stationary, as it corresponds to a sequence of ascending ideal sheaves $\{\mathcal{I}_j\}_j$ of $\mathcal{O}_{\widetilde{T_n}}$, starting with $\mathcal{I}_n = 0$. Call the limiting subscheme $\widetilde{T}$.

By construction, $\widetilde{T}$ has the desired universal property because a morphism $f : P \to T$ factors through $\widetilde{T}$ if and only if it factors through each of the inclusions in the sequence of nested subschemes $\widetilde{T}_j$, so if and only if each pullback $f^*\mathcal{E}_j$ is locally free on P. $\square$

Chapter 4

Hilbert and Quot schemes

4.1 Hilbert and Quot functors

Now we turn attention back to our original classification problem, which can be generalised in the following fruitful way. Consider for the rest of this chapter a noetherian scheme S, a projective S-scheme $\langle X, \pi \rangle$ and a coherent sheaf $\mathcal{E}$ on X. Fix also a relatively very ample line bundle $\mathcal{L}$ on X. Denote by $\underline{\mathcal{S}ch}_S$ the category of locally noetherian S-schemes.

Definition 4.1 *The* family of quotients functor $\mathfrak{Quot}_{\mathcal{E}/X/S} : \underline{\mathcal{S}ch}_S \to \mathcal{S}et$ *associates to each* $\langle T, \tau \rangle \in \underline{\mathcal{S}ch}_S$ *the set*

$$\left\{ (\mathcal{F}, q) \;\middle|\; \begin{array}{c} \mathcal{F} \text{ is a coherent } \mathcal{O}_{X_T}\text{-module flat over } T \\ q : \mathcal{E}_T \twoheadrightarrow \mathcal{F} \end{array} \right\} \Big/ \sim$$

under the equivalence relation of isomorphism as quotient sheaves of $\mathcal{E}_T$*, and to each morphism* $f : P \to T$ *the function* $\langle \mathcal{F}, q \rangle \mapsto \langle f_X^* \mathcal{F}, f_X^* q \rangle$*; where* $X_T = X \times_S T$ *and* $\mathcal{E}_T = \tau_X^* \mathcal{E}$*. We call a class in this set a* family of quotients of $\mathcal{E}$ parameterized by T *and denote it by* $\langle \mathcal{F}, q \rangle$*.*

Each scheme X_T inherits through the pullback of $\mathcal{L}$ by τ_X a relatively very ample line bundle, which we still denote by $\mathcal{L}$. In addition, we often abbreviate the spectrum of a ring $Spec\, A$ as A itself.

$$\begin{tikzcd}
Supp f_X^*\mathcal{F} \arrow[rrr, bend left] \arrow[dr, hook, "closed"] \arrow[dddr, "flat"'] & \mathcal{E}_P \arrow[r, two heads] \arrow[d, squiggly] & f_X^*\mathcal{F} \arrow[dl, squiggly] & Supp\mathcal{F} \arrow[dr, hook, "closed"] \arrow[dddr, "flat"'] & \mathcal{E}_T \arrow[r, two heads] \arrow[d, squiggly] & \mathcal{F} \arrow[dl, squiggly] \\
 & X_P \arrow[rrr, "f_X"] \arrow[drrrr, dotted] \arrow[dd, "\pi_P"'] & & & X_T \arrow[dr, dotted, "\tau_X"] \arrow[dd, "\pi_T"'] & \mathcal{E} \arrow[d, squiggly] \\
 & & & & & X \arrow[dd, "\pi"', "projective"] \\
 & P \arrow[rrr, "f"] \arrow[drrrr, dotted] & & & T \arrow[dr, dotted, "\tau"] & \\
 & & & & & S
\end{tikzcd}$$

Definition 4.2 *The* relative Hilbert functor $\mathfrak{Hilb}_{X/S} : \underline{\mathcal{S}ch}_S \to \mathcal{S}et$ *associates to each* S*-scheme* T *the set*

$$\{Y \subseteq X_T \mid Y \text{ is a family of subschemes of } \mathbb{P}^n_S \text{ parameterized by } T\}\,.$$

So we are considering families of subschemes of an arbitrary projective scheme X, and quotient sheaves of an arbitrary coherent sheaf $\mathcal{E}$. In order to approach the problem in a functorial way with respect to the parameterizing scheme T, it was necessary to fix X and $\mathcal{E}$ over a base scheme S, and then restricting to the category of S-schemes. The functors from Section 1.3 become $\mathfrak{Hilb}_{\mathbb{P}^n} = \mathfrak{Hilb}_{\mathbb{P}^n_{\mathbb{Z}}/\mathbb{Z}}$ and $\mathfrak{Quot}_{\mathbb{P}^n} = \mathfrak{Quot}_{\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}/\mathbb{P}^n_{\mathbb{Z}}/\mathbb{Z}}$, and we have the same isomorphism $\mathfrak{Hilb}_{X/S} \cong \mathfrak{Quot}_{\mathcal{O}_X/X/S}$ as in 1.15.

Analogously to the relative Grassmannian, $\mathfrak{Quot}_{\mathcal{E}/X/S}$ is a Zariski sheaf. This will again be crucial in the quest for representability because it allows us to reduce without loss of generality to an arbitrary open cover of the base scheme S. For example, since the coherent sheaves $\mathcal{F}$ considered above are flat over T, by 3.2 their Hilbert polynomial $\mathcal{H}^{\mathcal{L}}_{\mathcal{F},t}$ is locally constant as a function of $t \in T$, where we adopt the notation used in 3.1. So we can partition all the families of quotients by their Hilbert polynomials using the connected components of T.

Definition 4.3 *For all* $\Phi \in \mathbb{Q}[z]$*, the subfunctor* $\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ *associates to each* $\langle T, \tau\rangle \in \underline{\mathcal{S}ch}_S$ *the subset*

$$\{\langle \mathcal{F}, q\rangle \in \mathfrak{Quot}_{\mathcal{E}/X/S}(T) \mid \forall t \in T : \mathcal{H}^{\mathcal{L}}_{\mathcal{F},t} = \Phi\}\,.$$

Lemma 4.4 *There is an isomorphism of functors* $\mathfrak{Quot}_{\mathcal{E}/X/S} \cong \coprod_{\Phi \in \mathbb{Q}[z]} \mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$.

Proof Let $\langle T, \tau\rangle \in \underline{\mathcal{S}ch}_S$, and $\{U_i\}_i$ its cover of connected components. Then there is a sequence of natural isomorphisms represented in the following diagram. The two vertical arrows come from the equalizer sequence 1.4; they are isomorphisms because the U_i are disjoint and so the rightmost term in the sequence is a singleton. The horizontal equalities are due to the U_i being connected, since then the only coherent sheaves $\mathcal{F}$ on T_i are the ones with a constant Hilbert polynomial, by 3.2.

$$\begin{tikzcd}
\prod_i \mathfrak{Quot}_{\mathcal{E}/X/S}(U_i) \arrow[r, phantom, "="] & \prod_i \coprod_{\Phi \in \mathbb{Q}[z]} \left(\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}(U_i)\right) \arrow[r, phantom, "="] & \prod_i \left(\coprod_{\Phi \in \mathbb{Q}[z]} \mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}\right)(U_i) \\
\mathfrak{Quot}_{\mathcal{E}/X/S}(T) \arrow[u, "\wr"] & & \left(\coprod_{\Phi \in \mathbb{Q}[z]} \mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}\right)(T) \arrow[u, "\wr"]
\end{tikzcd}$$

$\square$

In our primary reference, the discussion above holds more generally for an S-scheme X which is merely of finite type over S; but that requires generalizing accordingly our results about flatness and Hilbert polynomials in a way outside the scope of this paper. Moreover, in that case the family of quotients functor is not in general representable. Assuming that X is projective is enough to prove that each of the subfunctors $\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ is representable by locally noetherian S-schemes, implying that $\mathfrak{Quot}_{\mathcal{E}/X/S}$ is representable by their scheme-theoretic coproduct.

$$Quot_{\mathcal{E}/X/S} \cong \coprod_{\Phi\in\mathbb{Q}[z]} Quot^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$$

Having assumed that X is projective over S, there is a closed embedding $\iota : X \hookrightarrow \mathbb{P}_S(\mathcal{V})$ into some projective bundle, which on small enough open subsets $U \subseteq S$ extends to closed embeddings $X_U \hookrightarrow \mathbb{P}^n_U$ into some projective space. In our quest for representability, with the following lemma we will reduce to the case $X = \mathbb{P}_S(\mathcal{V})$. Through gluing, even a reduction to the case $X = \mathbb{P}^n_S$ is possible, but it won't yield the full result we are aiming for; this will be discussed after the proof of the main existence theorem 4.14.

Lemma 4.5 *For all $\Phi \in \mathbb{Q}[z]$, any closed embedding $\iota : X \hookrightarrow Y$ of projective S-schemes and any relatively very ample line bundles $\mathcal{L}'$ on Y with $\mathcal{L} = \iota^*\mathcal{L}'$, pushing forward by ι induces an isomorphism of functors $\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S} \cong \mathfrak{Quot}^{\Phi,\mathcal{L}'}_{\iota_*\mathcal{E}/Y/S}$.*

Proof Let $\langle T, \tau\rangle \in \underline{\mathcal{S}ch}_S$. Denote by $\iota_T : X_T \hookrightarrow Y_T$ the base extension of ι by τ, which is still a closed embedding. The direct image functor ι_{T*} is a category equivalence between coherent sheaves on X_T and coherent sheaves on Y_T supported in X_T. Being an exact functor, it maps quotient sheaves of $\mathcal{E}_T$ to quotient sheaves of $\iota_{T*}\mathcal{E}_T$. In addition, this equivalence is natural in T, respects the equivalence relation in $\mathfrak{Quot}_{\mathcal{E}/X/S}(T)$ and preserves flatness over T. It also preserves the Hilbert polynomial, as cohomology in a closed subscheme can be also computed in the ambient scheme through the direct image [Har77, III.2.10]. □

Lemma 4.6 *For all $\Phi \in \mathbb{Q}[z]$ and $\lambda \in \mathbb{Z}$, tensoring by $\mathcal{L}^{\otimes\lambda}$ induces an isomorphism of functors $\mathfrak{Quot}^{\Phi(z),\mathcal{L}}_{\mathcal{E}/X/S} \cong \mathfrak{Quot}^{\Phi(z+\lambda),\mathcal{L}}_{\mathcal{E}(\lambda)/X/S}$.*

Proof Tensoring by $\mathcal{L}$ commutes with the pullback, which is used to determine the Hilbert polynomial of the sheaves $\mathcal{F}|_{X_t}$ on the fibers X_t of X_T. Thus, it holds $\mathcal{H}^{\mathcal{L}}_{\mathcal{F}(\lambda),t}(z) = \mathcal{H}^{\mathcal{L}}_{\mathcal{F},t}(z+\lambda)$. □

4.2 Grassmannians as Quot Schemes

Proposition 4.7 *In the case $X = S$, for every $d \in \mathbb{N}$ there is an isomorphism of functors $\mathfrak{Grass}_S(\mathcal{E}, d) \cong \mathfrak{Quot}^{d,\mathcal{O}_S}_{\mathcal{E}/S/S}$.*

Proof Since the contravariant relation of both functors is the same, we just have to verify for an arbitrary $T \in \underline{\mathcal{S}ch}_S$ that $\mathfrak{Grass}_S(\mathcal{E}, d)(T) = \mathfrak{Quot}^{d,\mathcal{O}_S}_{\mathcal{E}/S/S}(T)$. Indeed, as we have seen in chapter 3.1, a coherent sheaf on T is locally free if and only if it is flat over T; furthermore, in this case, its rank is precisely its Hilbert polynomial at every point of $t \in T$. $\square$

As a consequence, since any family of quotients in $\mathfrak{Quot}_{\mathcal{E}/S/S}$ has at each $s \in S$ a degree 0 Hilbert polynomial, in this case the relative Grassmannians are all that we actually need to represent the family of quotients.

$$Quot_{\mathcal{E}/S/S} \cong \coprod_{d \in \mathbb{N}} Grass_S(\mathcal{E}, d)$$

For the rest of this section, consider the motivating case $X = \mathbb{P}^n_S$ discussed in the introduction 1.3, with $\mathcal{L} = \mathcal{O}(1)$. As a starting point, we restrict attention to families of subschemes $Y \subseteq \mathbb{P}^n_T$ with Hilbert polynomial $\Phi = 1$ everywhere over T, for some $T \in \underline{\mathcal{S}ch}_S$. Then, each fiber Y_t is actually isomorphic to $Spec\, k(t)$, and the flatness condition translates intuitively to Y_t varying as a subscheme of Y in the same way that $Spec\, k(t)$ varies as a subscheme of T; which is a clue that $\pi_Y : Y \to T$ should be an isomorphism.

If this is the case, T would come equipped with the unique morphism $f = \tau_{\mathbb{P}} \circ \pi_Y^{-1} : T \to \mathbb{P}^n_S$ of S-schemes such that $\pi_{Y*}\mathcal{O}_Y(1) \cong f^*\mathcal{O}(1)$, through an isomophism of line bundles that is consistent with the respective $n + 1$ global sections. Indeed, $\pi_{Y*}\mathcal{O}_Y(1)$ is a line bundle on T by Lemma 3.3, because $\mathcal{O}_Y(1)$ is flat, has Serre vanishing and its Hilbert polynomial is 1; so there exists such an f by the universal property of $\mathbb{P}^n_S$.

By the universal property of the fibered product, f lifts to a morphism $\widetilde{f} : T \to \mathbb{P}^n_T$ of S-schemes with right inverse π_T; this is a proper monomorphism, i.e. a closed embedding. Moreover, $\iota = \widetilde{f} \circ \pi_Y$ by the universal property of $\mathbb{P}^n_T$, because they both pull back the twisting sheaf to the same line bundle, namely $\mathcal{O}_Y(1)$. $\widetilde{f}$ factors through ι to an inverse morphism of π_Y because $\iota_*\mathcal{O}_Y = \widetilde{f}_*\pi_{Y*}\mathcal{O}_Y = \widetilde{f}_*\mathcal{O}_T$.

$$\begin{tikzcd}
\mathbb{P}^n_T \arrow[r, "\tau_{\mathbb{P}}"] \arrow[dd, bend right, "\pi_T"'] & \mathbb{P}^n_S \arrow[dd, "\pi_S"] \\
Y \arrow[u, hook] \arrow[d, "\pi_Y"] & \\
T \arrow[r, "\tau"] \arrow[uu, dashed, bend right, "\widetilde{f}"'] \arrow[uur, dashed, "f"] & S
\end{tikzcd}$$

Summarizing, we found an isomorphism $\mathfrak{Hilb}^{1,\mathcal{O}(1)}_{\mathbb{P}^n_S/S} \cong \mathfrak{Grass}_S(n+1, 1)$, so the Hilbert scheme $Hilb^{1,\mathcal{O}(1)}_{\mathbb{P}^n_S/S}$ is $\mathbb{P}^n_S$ itself. This is the simplest example of a

Hilbert scheme of d-points $Hilb^{d,\mathcal{O}(1)}_{\mathbb{P}^n_S/S}$, named after the classification problem it represents: families of d points on $\mathbb{P}^n_S$ parameterized by T; but we will not discuss this in more detail here.

By retracing our steps, we also recognize that the inverse natural transformation sends a line bundle $\mathcal{O}_T^{\oplus n+1} \twoheadrightarrow \mathcal{E}$ to the projective bundle $\mathbb{P}_T(\mathcal{E})$, which is a closed subscheme of $\mathbb{P}^n_T$ by Lemma 2.7. The discussion above holds more generally for quotient bundles of any rank. In the next proposition we will verify that indeed every closed subscheme of $\mathbb{P}^n_T$ which is flat over T with Hilbert polynomial $\Phi_d(z) = \binom{z+d}{d}$ arises as the projective bundle $\mathbb{P}_T(\mathcal{E})$ for some rank $d+1$ quotient bundle of $\mathcal{O}_T^{\oplus n+1}$. Thus, the relative Grassmannian scheme $Grass_S(n+1, d+1)$ parameterizes all the families of subschemes of $\mathbb{P}^n_S$ which have Hilbert polynomial Φ_d. But first, we have to study the nature of the closed embedding in more detail.

Let T be a locally noetherian scheme, $\sigma : \mathcal{E}' \twoheadrightarrow \mathcal{E}$ a surjection of coherent sheaves of $\mathcal{O}_T$-modules and $\iota : Y = \mathbb{P}_T(\mathcal{E}) \hookrightarrow \mathbb{P}_T(\mathcal{E}')$ the closed embedding given by Lemma 2.7. Recall that any projective bundle $\mathbb{P}_T(\mathcal{E})$ projects through π_{T*} its twisting sheaf $\mathcal{O}_{\mathbb{P}_T(\mathcal{E})}(1)$ precisely to $\mathcal{E}$ [Har77, II.7.11]. Now we adapt the diagram seen in the proof of 2.7, which illustrates the universal property of $\mathbb{P}_T(\mathcal{E})$ as a subscheme of $\mathbb{P}_T(\mathcal{E}')$. Namely, we look at how the SES of σ, which is the projection through π_{T*} of the SES $\mathcal{G} \hookrightarrow \mathcal{O}(1) \twoheadrightarrow \iota_*\mathcal{O}_Y(1)$, behaves under pullback through the commutative diagram $\pi_Y = \pi_T \circ \iota$.

$$\begin{tikzcd}
\pi_Y^*\mathcal{K} \arrow[r, "\pi_Y^*\kappa"] \arrow[dr, "\iota^*\gamma=0"'] & \pi_Y^*\mathcal{E}' \arrow[r, two heads, "\pi_Y^*\sigma"] \arrow[d, two heads, "\iota^*q'"] & \pi_Y^*\mathcal{E} \arrow[dl, two heads, "q"] \\
 & \mathcal{O}_Y(1) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_T^*\pi_{T*}\mathcal{G} \arrow[r, "\pi_T^*\kappa"] \arrow[d, two heads, "ad"] \arrow[dr, "\gamma"] & \pi_T^*\mathcal{E}' \arrow[r, two heads, "\pi_T^*\sigma"] \arrow[d, two heads, "ad=q'"] & \pi_T^*\mathcal{E} \arrow[r, squiggly, -|] \arrow[d, two heads, "ad=\iota_*q"] & \mathbb{P}_T(\mathcal{E}') \arrow[dd, bend left, "\pi_T"] \\
\mathcal{G} \arrow[r, hook] & \mathcal{O}(1) \arrow[r, two heads] & \iota_*\mathcal{O}_Y(1) \arrow[ur, squiggly, -|] & \mathbb{P}_T(\mathcal{E}) \arrow[u, hook, "\iota"] \arrow[d, "\pi_Y"'] \\
\pi_{T*}\mathcal{G} \arrow[r, hook, "\kappa"] & \mathcal{E}' \arrow[r, two heads, "\sigma"] & \mathcal{E} \arrow[r, squiggly, -|] & T
\end{tikzcd}$$

The middle term of each row is exact in its sequence. The vertical maps *ad* correspond to the identity through the π_{T*}/π_T^* adjointness property, they are surjective because the sheaves $\mathcal{G}$, $\mathcal{O}(1)$ and $\iota_*\mathcal{O}_Y(1)$ are relatively globally generated, as pointed out in [Har77, III.8.8]. For the two rightmost *ad* maps this is actually obvious, after noticing that they are respectively the tautological bundle q' on $\mathbb{P}_T(\mathcal{E}')$ and the direct image by ι_* of the tautological bundle q on $\mathbb{P}_T(\mathcal{E})$.

Then we recognize $\iota_*\mathcal{O}_Y(1)$ as precisely the cokernel of the composition $\gamma = q' \circ \pi_T^*\kappa : \pi_T^*\pi_{T*}\mathcal{G} \to \mathcal{O}(1)$. This means that the closed subscheme $\mathbb{P}_T(\mathcal{E}) \hookrightarrow \mathbb{P}_T(\mathcal{E}')$ is determined as the schematic support of the coherent quotient sheaf $\mathcal{O}_{\mathbb{P}_T(\mathcal{E}')} \twoheadrightarrow coker(\gamma) \otimes \mathcal{O}(-1)$.

Proposition 4.8 *For every $n \geq d \in \mathbb{N}$, projecting the twisting sheaf through π with its $n+1$ global sections induces an isomorphism of functors $\mathfrak{Hilb}^{\Phi_d,\mathcal{O}(1)}_{\mathbb{P}^n_S/S} \cong \mathfrak{Grass}_S(n+1,d+1)$, where $\Phi_d(z) = \binom{z+d}{d}$.*

Proof First we check that taking the projective bundle, with its closed embedding given by Lemma 2.7, gives a well-defined natural transformation

$$\mathbb{P}:\ \mathfrak{Grass}_S(n+1,d+1) \to \mathfrak{Hilb}^{\Phi_d,\mathcal{O}(1)}_{\mathbb{P}^n_S/S}\ .$$

Let $\langle \mathcal{J}, j\rangle \in \mathfrak{Grass}_S(n+1,d+1)(T)$ for some $T \in \underline{\mathcal{S}ch}_S$. The naturality of $\mathbb{P}$ is just a rephrasing of the base change property 2.6. We can use naturality to focus on trivializing neighbourhoods $U \subseteq T$ for $\mathcal{J}$, where we recognize $\mathbb{P}_U(\mathcal{E}|_U) \cong \mathbb{P}^d_U$ as a flat U-scheme, whose Hilbert polynomial is known to be Φ_d everywhere over U.

Conversely, given a subscheme $Y \in \mathfrak{Hilb}^{\Phi_d,\mathcal{O}(1)}_{\mathbb{P}^n_S/S}(T)$, we can apply 3.3 to the sheaf $\mathcal{O}_Y(1)$, which is flat over T by assumption and satisfies Serre vanishing. So the projection of $\mathcal{O}(1) \twoheadrightarrow \iota_*\mathcal{O}_Y(1)$ to T is the rank $\Phi_d(1) = d+1$ quotient bundle

$$j:\ \mathcal{O}_T^{\oplus n+1} = \pi_{T*}\mathcal{O}(1) \twoheadrightarrow \pi_{Y*}\mathcal{O}_Y(1) = \mathcal{J}\ .$$

The direct image by ι_* of the sheaves on Y fit into the same diagram shown above that characterizes the subscheme $\mathbb{P}_T(\mathcal{J}) \hookrightarrow \mathbb{P}_T(\mathcal{O}_T^{\oplus n+1}) = \mathbb{P}^n_T$. So the structure sheaves of Y and $\mathbb{P}_T(\mathcal{J})$ on $\mathbb{P}^n_T$ are isomorphic as quotients sheaves of $\mathcal{O}_{\mathbb{P}^n_T}$, i.e. they are the same subscheme by A.1. $\square$

To prove representability of the general family of quotients functor we will find a natural transformation into a relative Grassmannian analogous to the one above, but we have to handle three complications.

Firstly, to define an element of a Grassmannian functor we need a quotient of a pulled back sheaf from S. The obvious choice is $\tau^*\pi_*\mathcal{E}$ because we can precompose the projection through π_T of a quotient of $\mathcal{E}_T$ by the base-change morphism $\tau^*\pi_*\mathcal{E} \to \pi_{T*}\mathcal{E}_T$.

$$\begin{tikzcd}
\mathcal{E}_T \arrow[r, two heads, "q"] & \mathcal{F} \arrow[r, squiggly] & X_T \arrow[d, "\pi_T"] \\
\tau^*\pi_*\mathcal{E} \arrow[r, two heads] & \pi_{T*}\mathcal{F} \arrow[r, squiggly] & T
\end{tikzcd}$$

Secondly, $\pi_{T*}\mathcal{F}$ is not in general a locally free sheaf of constant rank assuming just its flatness over T. We do know that a high enough twist, by a Serre vanishing lower bound N, will make it into one by Lemma 3.3, and since we can focus on families of constant Hilbert polynomial Φ by Lemma 4.4, the resulting sheaf $\pi_{T*}\mathcal{F}(N)$ will be a rank $\Phi(N)$ vector bundle. The problem is that N depends on $\mathcal{F}$, so we can not a priori choose a maximal one that works for every family $\mathcal{F}$. Nonetheless, this is actually possible because of the following facts.

- Since $\mathcal{F}$ is flat, $\mathcal{F}(z)$ has Serre vanishing if and only if it has vanishing higher cohomology on each fiber X_t of T.
- Since S is noetherian, each fiber X_t will embed into $\mathbb{P}^n_{k(t)}$ and each $\mathcal{F}|_t$ will admit a surjection from $\mathcal{O}^{\oplus p}_{\mathbb{P}^n_{k(t)}}$, for some n, p independent of T, $\mathcal{F}$.
- There is a lower bound m such that, for any field k, all the quotient sheaves of $\mathcal{O}^{\oplus p}_{\mathbb{P}^n_k}$ on $\mathbb{P}^n_k$ with fixed Hilbert polynomial Φ have vanishing higher cohomology.

We prove the existence of such a lower bound m in the next section.

One can still pull back any quotient bundle $\langle \mathcal{J}, j\rangle$ of $\tau^*\pi_*\mathcal{E}$ to X_T and then take the cokernel of the map γ, as called in a preceding diagram. But the respective quotient sheaf $\mathcal{E}_T \twoheadrightarrow \beta(\langle \mathcal{J}, j\rangle)$ it induces won't in general be flat over T.

$$\begin{tikzcd}
\mathcal{E}_T \arrow[r, two heads] & \beta(\langle \mathcal{J}, j\rangle) \arrow[r, rightsquigarrow] & X_T \arrow[d, "\pi_T"] \\
\tau^*\pi_*\mathcal{E} \arrow[r, two heads, "j"] & \mathcal{J} \arrow[r, rightsquigarrow] & T
\end{tikzcd}$$

So the third complication lies in the fact there is not an inverse transformation to the family of quotients functor. To determine the image of $\mathfrak{Q}uot$ into $\mathfrak{G}rass$ we will need a criterion for when the pullback $f^*\langle \mathcal{J}, j\rangle$ does indeed define a quotient $\beta(f^*\langle \mathcal{J}, j\rangle)$ flat over P, along an arbitrary morphism $f : P \to T$. It is here that the flattening stratification theorem will come into play, delivering a locally closed subscheme T_Φ of T which determines this criterion.

4.3 Castelnuovo-Mumford regularity

In this section we present the content covered in [Mum66, §14]. Let $n \in \mathbb{N}$ and k be any field. We consider a coherent sheaf $\mathcal{G}$ on projective space $\mathbb{P}^n_k$.

Definition 4.9 *For $m \in \mathbb{Z}$, $\mathcal{G}$ is called* m-regular *if*

$$\forall i \geq 1 : \quad H^i(\mathbb{P}^n_k, \mathcal{G}(m-i)) = 0\,.$$

For example, consider the twisting sheaf $\mathcal{O}_{\mathbb{P}^n_k}(r)$ for $r \in \mathbb{Z}$, which is m-regular if and only if $H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(r+m-n)) = 0$ since all other higher cohomology groups vanish in any case. This cohomology group is known to be $\left(\frac{1}{x_0 \cdots x_n} k[x_0 \cdots x_n]\right)_{r+m-n}$ for $r+m-n < 0$, and 0 otherwise. So $\mathcal{O}_{\mathbb{P}^n_k}(r)$ is m-regular if and only if $m \geq -r$.

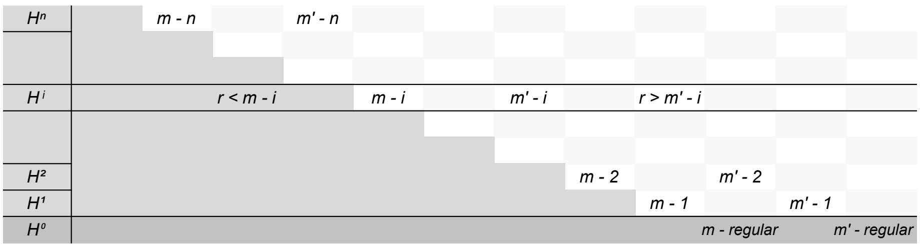
The following lemma, that Mumford attributed to Castelnuovo, shows how m-regularity lower bounds are useful to annihilate all the higher cohomology groups by means of tensoring with the twisting sheaf.

Lemma 4.10 (Castelnuovo) *If $\mathcal{G}$ is m-regular, then the following statements hold.*

a) For all $m' \geq m$, $\mathcal{G}$ is also m'-regular. In particular, for all $r \geq m$, $\mathcal{G}(r)$ has vanishing higher cohomology.

b) The canonical map $\Gamma(\mathbb{P}^n_k, \mathcal{G}(r)) \otimes \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1)) \xrightarrow{\mu} \Gamma(\mathbb{P}^n_k, \mathcal{G}(r+1)$ is surjective for all $r \geq m$.

c) For all $r \geq m$, $\mathcal{G}(r)$ is generated by its global sections.



Proof The lemma is proved by induction on n. The case $n = 0$ is readily verified. We will restrict $\mathcal{G}$ to a hyperplane in $\mathbb{P}^n_k$, isomorphic to $\mathbb{P}^{n-1}_k$, which should not contain associated points of $\mathcal{G}$ for the induction step to work. Recalling that a coherent sheaf on a noetherian scheme has only finitely many associated points, the existence of such a hyperplane would be certain if the base field k were infinite. We can assume this without loss of generality by base-changing under an infinite field extension $k' \to k$. Then, since $Spec\, k' \to Spec\, k$ is a flat morphism of schemes, we can apply [Har77, III.9.3] to get isomorphisms between the cohomologies of $\mathcal{G}$ on $\mathbb{P}^n_k$ and its pullback $\mathcal{G}'$ on $\mathbb{P}^n_{k'}$.

So assume the lemma holds for the case $n-1$. Take a hyperplane $H = V_+(h) \subseteq \mathbb{P}^n_k$, for some homogeneous equation h, that doesn't contain any associated point of $\mathcal{G}$. Then, for all $\mathfrak{p} \in \mathbb{P}^n_k$, the map of $\mathcal{O}_{\mathfrak{p},\mathbb{P}^n_k}$-modules $g \in \mathcal{G}_\mathfrak{p} \mapsto g \otimes \overline{h} \in \mathcal{G}_\mathfrak{p} \otimes \mathcal{O}(1)_\mathfrak{p}$ is injective, so we obtain the exact sequence

$$0 \to \mathcal{G}(m-i-1) \xrightarrow{\otimes h} \mathcal{G}(m-i) \to \mathcal{G}_H(m-i) \to 0$$

for all $i \geq 1$, where $\mathcal{G}_H = \iota_*\iota^*\mathcal{G}$ for $\iota : H \hookrightarrow \mathbb{P}^n_k$ the closed embedding. Notice that $H^i(\mathbb{P}^n_k, \mathcal{G}_H(r)) = H^i(H, \iota^*\mathcal{G}_H(r))$, so in this proof we will use these terms interchangeably. Since $\mathcal{G}$ is m-regular, by looking at the long exact sequence in cohomology we see that also $\iota^*\mathcal{G}_H$ is m-regular, and thus it satisfies the claims of the lemma by the inductive hypothesis.

Now we verify claim $a)$ for $\mathcal{G}$ by showing $H^i(\mathbb{P}^n_k, \mathcal{G}(r)) = 0$ for all $i \geq 1$, $r \geq m-i$ using induction on r. In the case $r = m-i$, $H^i(\mathbb{P}^n_k, \mathcal{G}(r)) = 0$ because $\mathcal{G}$ is m-regular. For $r > m-i$, focusing on the i^{th} excerpt from the LES in cohomology, we see that $H^i(\mathbb{P}^n_k, \mathcal{G}(r)) = 0$ holds since the first term is zero by the inductive hypothesis on r, and the right term is zero because $\iota^*\mathcal{G}_H$ satisfies claim $a)$.

$$H^i(\mathbb{P}^n_k, \mathcal{G}(r-1)) \to H^i(\mathbb{P}^n_k, \mathcal{G}(r)) \to H^i(\mathbb{P}^n_k, \mathcal{G}_H(r)) = H^i(H, \iota^*\mathcal{G}_H(r))$$

For claim $b)$, let $r \geq m$. The multiplication map in the claim fits into a diagram where the bottom row is the 0^{th} excerpt from the long exact sequence in cohomology.

$$\begin{tikzcd}
 & \Gamma(\mathbb{P}^n_k, \mathcal{G}(r)) \otimes \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1)) \arrow[r, two heads] \arrow[d, "\mu"] & \Gamma(H, \iota^*\mathcal{G}_H(r)) \otimes \Gamma(H, \mathcal{O}_H(1)) \arrow[d, two heads, "\mu_H"] \\
\Gamma(\mathbb{P}^n_k, \mathcal{G}(r)) \arrow[ur, "\otimes h"] \arrow[r, hook, "\otimes h"] & \Gamma(\mathbb{P}^n_k, \mathcal{G}(r+1)) \arrow[r, two heads, "\rho"] & \Gamma(H, \iota^*\mathcal{G}_H(r+1))
\end{tikzcd}$$

$\mathcal{G}$ satisfies claim $a)$ and $r-1 \geq m-1$, so we have $H^1(\mathbb{P}^n_k, \mathcal{G}(r-1)) = 0$. This implies that the top map is surjective, being the tensor product of two surjective maps. The map μ_H is surjective because $\iota^*\mathcal{G}_H$ satisfies claim $b)$ by induction. The diagram is commutative because the tensor product commutes with cohomology in a natural way, so the composition $\rho \circ \mu$ is surjective. Therefore we obtain $\Gamma(\mathbb{P}^n_k, \mathcal{G}(r+1)) = im(\mu) + ker(\rho) = im(\mu) + im(\otimes h) = im(\mu)$, i.e. μ is surjective.

For $p >> 0$ we know that $\mathcal{G}(r+p)$ is generated by global sections, as $\mathcal{O}_{\mathbb{P}^n_k}(1)$ is ample. By repeated application of claim $b)$ for $\mathcal{G}$, we obtain that $\Gamma(\mathbb{P}^n_k, \mathcal{G}(r)) \otimes \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(p)) \xrightarrow{\mu} \Gamma(\mathbb{P}^n_k, \mathcal{G}(r+p))$ is surjective, so we can choose a generating set of sections $(g_i \otimes y_i)_i$ for $\mathcal{G}(r+p)$, where $g_i \in \Gamma(\mathbb{P}^n_k, \mathcal{G}(r))$, $y_i \in \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(p))$. Then $\mathcal{G}(r)$ is also generated by global sections, namely from $(f_i)_i$; thus claim $c)$ has also been verified. $\square$

Lemma 4.11 *Suppose $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is an exact sequence of coherent sheaves on $\mathbb{P}^n_k$. If any two terms in the sequence are m-regular, then the third is at least $m+1$-regular.*

Just by knowing the Hilbert polynomial of a coherent sheaf $\mathcal{F}$ on some projective space $\mathbb{P}^n_T$, the following theorem is what will allow us to deduce the regularity properties above for all the fibers $\mathcal{F}|_t$ on $\mathbb{P}^n_{k(t)}$ indipendently of $\mathcal{F}$ and the point t.

Theorem 4.12 (Mumford) *For every $p, n \in \mathbb{N}$, there exists a polynomial $F_{p,n}$ in $n+1$ varibles with integral coefficients, which has the following property:*

> *Let $\mathcal{G}$ be any coherent subsheaf of $\mathcal{O}^{\oplus p}_{\mathbb{P}^n_k}$ on $\mathbb{P}^n_k$ over any field k, whose Hilbert polynomial has decomposition in binomial coefficients as*
>
> $$\mathcal{H}^{\mathcal{O}(1)}_{\mathcal{G}}(z) = \sum_{i=0}^{n} a_i \binom{z}{i},$$
>
> *where $(a_i)_{0 \le i \le n} \in \mathbb{Z}$. Then $\mathcal{G}$ is $F_{p,n}((a_i)_{0 \le i \le n})$-regular.*

For our purposes, we only need an m-regularity lower bound for all sheaves with a Hilbert polynomial Φ fixed beforehand, and we are not interested in the polynomial relation $F_{p,n}$ between Φ and the m-regularity. So we prove the following special case of Mumford's theorem, which will allow us to simplify slightly the proof by ignoring all the combinatorial details of binomial coefficients.

Theorem 4.13 *For every $p, n \in \mathbb{Z}$ and $\Phi \in \mathbb{Q}[z]$, there exists an $m \in \mathbb{Z}$ such that for every coherent subsheaf or quotient sheaf $\mathcal{G}$ of $\mathcal{O}^{\oplus p}_{\mathbb{P}^n_k}$ on $\mathbb{P}^n_k$ over any field k with Hilbert polynomial $\mathcal{H}^{\mathcal{O}(1)}_{\mathcal{G}}(z) = \Phi$, $\mathcal{G}$ is m-regular.*

Proof First note that quotient sheaves of $\mathcal{O}^{\oplus p}_{\mathbb{P}^n_k}$ with polynomial Φ correspond through their SES to subsheaves with polynomial $p - \Phi$. By taking the bigger m of the two, in light of Lemma 4.11 it is enough to prove the claim just for subsheaves.

We prove the theorem by induction on n. The case $n = 0$ is trivial, so assume the theorem holds for $n-1$, and consider any coherent subsheaf $\mathcal{G}$ of $\mathcal{O}^{\oplus p}_{\mathbb{P}^n_k}$ with Hilbert polynomial Φ. As in the proof of 4.10, we can assume k to be infinite in order to find a hyperplane $H \subseteq \mathbb{P}^n_k$ which doesn't contain any associated point of the coherent sheaf $\mathcal{O}^{\oplus p}_{\mathbb{P}^n_k}/\mathcal{G}$. Then the torsion sheaf $Tor_1(\mathcal{O}_H, \mathcal{O}^{\oplus p}_{\mathbb{P}^n_k}/\mathcal{G})$ vanishes, meaning that tensoring by $\mathcal{O}_H = \iota_* \iota^* \mathcal{O}_{\mathbb{P}^n_k}$ preserves exactness of $\mathcal{G} \hookrightarrow \mathcal{O}^{\oplus p}_{\mathbb{P}^n_k}$, yielding the restricted short exact sequence

$$0 \to \mathcal{G}_H \to \mathcal{O}^{\oplus p}_H \to \mathcal{O}^{\oplus p}_H / \mathcal{G}_H \to 0\,.$$

Thus, $\mathcal{G}_H$ is a subsheaf of $\mathcal{O}_H^{\oplus p}$, where $H \cong \mathbb{P}_k^{n-1}$. As the Hilbert polynomial is additive on short exact sequences, from $0 \to \mathcal{G}(-1) \to \mathcal{G} \to \mathcal{G}_H \to 0$ we can note $\mathcal{H}_{\mathcal{G}_H}^{\mathcal{O}(1)}(z) = \Phi(z) - \Phi(z-1)$. So from the induction hypothesis $\mathcal{G}_H$ is m_0-regular for some $m_0 \in \mathbb{Z}$ depending only on Φ.

Now we search some $m \in \mathbb{Z}$ depending on m_0 for which $\mathcal{G}$ is m-regular. Consider the long exact sequence in cohomology of $0 \to \mathcal{G}(m-1) \to \mathcal{G}(m) \to \mathcal{G}_H(m) \to 0$, for $m \geq m_0 - 1$.

$$0 \to \Gamma(\mathcal{G}(m-1)) \to \Gamma(\mathcal{G}(m)) \xrightarrow{\rho_m} \Gamma(\mathcal{G}_H(m)) \to H^1(\mathcal{G}(m-1)) \xrightarrow{\alpha} H^1(\mathcal{G}(m)) \to 0 \tag{4.1}$$

$$i \geq 2: \qquad 0 \to H^i(\mathcal{G}(m-1)) \xrightarrow{\sim} H^i(\mathcal{G}(m)) \to 0 \tag{4.2}$$

Since we have $H^i(\mathcal{G}_H(m)) = 0$ for all $i \geq 2$ and $m >> 0$, by 4.2 this also holds for all $m \geq m_0 - 2$. So we only need to focus on annihilating $H^1(\mathcal{G}(m-1))$. Since α is surjective, the function $h^1(\mathcal{G}(m)) = dim_k H^1(\mathcal{G}(m))$ is decreasing for all $m \geq m_0$. In addition, $h^1(\mathcal{G}(m))$ eventually reaches 0, so it cannot be constant. In fact this function is strictly decreasing because of the following contraposition.

If $h^1(\mathcal{G}(m-1)) = h^1(\mathcal{G}(m))$ for some $m \geq m_0$, then, by exactness of 4.1 at $H^1(\mathcal{G}(m-1))$, ρ_m would be surjective. Since $\mathcal{G}_H$ is m_0-regular, the map μ_H in the diagram for the proof of 4.10.2) is surjective. So the map $\rho_{m+1} : \Gamma(\mathcal{G}(m+1)) \to \Gamma(\mathcal{G}_H(m+1))$ in that diagram will also be surjective. Implying in turn that $h^1(\mathcal{G}(m)) = h^1(\mathcal{G}(m+1))$, but $m \geq m_0$ was arbitrary so this cannot be because $h^1(\mathcal{G}(m))$ eventually reaches 0. So we conclude that $h^1(\mathcal{G}(m))$ is strictly decreasing.

This means we can bound the amount of steps it takes for $h^1(\mathcal{G}(m))$ to reach 0, starting from m_0, by $h^1(\mathcal{G}(m_0))$ itself. So we get that $H^1(\mathcal{G}(m)) = 0$ for all $m \geq m_0 + h^1(\mathcal{G}(m_0))$.

Now we bound $h^1(\mathcal{G}(m_0))$ indipendently of $\mathcal{G}$. As $\mathcal{G} \hookrightarrow \mathcal{O}_{\mathbb{P}_k^n}^{\oplus p}$, we must have $dim_k \Gamma(G(m_0)) \leq p \cdot dim_k \Gamma(\mathcal{O}_{\mathbb{P}_k^n}(m_0)) = p\binom{n+m_0}{n}$. Since for all $i \geq 2$ we have checked $H^i(\mathcal{G}(m_0)) = 0$, the Hilbert polynomial Φ reduces to

$$\Phi(m_0) = dim_k \Gamma(G(m_0)) - h^1(\mathcal{G}(m_0))\,,$$

$$h^1(\mathcal{G}(m_0)) \leq p\binom{n+m_0}{n} - \Phi(m_0)\,.$$

In conclusion, for $m \geq m_0 + p\binom{n+m_0}{n} - \Phi(m_0)$, every such sheaf $\mathcal{G}$ will be m-regular. $\square$

4.4 Main existence theorem

In this section we will finally prove that the general functor $\mathfrak{Quot}_{\mathcal{E}/X/S}$ is representable. Our proof is an adaptation of the one presented in [Nit05, 5.2], which goes back to Altman and Kleiman [AK80, 2.6]. In that proof, some additional assumptions are made about the nature of X and $\mathcal{E}$, which we discuss in the next section, and afterwards it is sketched how to conclude the general case. In our version, we prove directly the general case, as stated in Grothendieck's original theorem [Nit05, 5.1].

Theorem 4.14 *For every $\Phi \in \mathbb{Q}[z]$, the family of quotients functor $\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ is representable by a projective locally noetherian scheme $Quot^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ over S.*

With previous results we have reduced without loss of generality to proving the theorem for $\mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S}$ in the following context: X is a projective bundle $\mathbb{P}_S(\mathcal{V})$ of a coherent sheaf $\mathcal{V}$ on S (by 4.5) and $\mathcal{E}$ is a globally generated coherent sheaf on X (by 4.6).

Proof **Projecting families on X_T to locally free sheaves on T**

Having assumed that $\mathcal{E}$ is globally generated, it admits a surjection from $\mathcal{O}_X^{\oplus p}$ for some p. As S is noetherian, it has a finite open cover $\{U_i\}_i$ of subsets on which $\mathcal{V}$ is globally generated. Then, any surjection $\mathcal{O}_{U_i}^{\oplus n} \twoheadrightarrow \mathcal{V}|_{U_i}$ yields by 2.7 a closed embedding $\mathbb{P}_{U_i}(\mathcal{V}|_{U_i}) \hookrightarrow \mathbb{P}^n_{U_i}$, where n can be taken big enough to work for all i. So for each $s \in S$, there is a closed embedding $\iota_s : X_s = \mathbb{P}_S(\mathcal{V})_s \hookrightarrow \mathbb{P}^n_{k(s)}$. Moreover, for every S-scheme $\langle T, \tau \rangle$, the base extension of $\iota_{\tau(t)}$ by τ gives closed embeddings $\iota_t : X_t = \mathbb{P}_T(\tau^*\mathcal{V})_t \hookrightarrow \mathbb{P}^n_{k(t)}$ for every $t \in T$.

Now consider $T \in \underline{\mathcal{S}ch}_S$ and any coherent quotient sheaf $q : \mathcal{E}_T \twoheadrightarrow \mathcal{F}$ on X_T, which has Hilbert polynomial Φ everywhere over T. In order to transport families of quotients on X_T to locally free sheaves on T we use Lemma 3.3, but we need a lower bound N for Serre vanishing indipendent of the actual sheaf. Since all the families we are interested in have the same Hilbert polynomial Φ everywhere, an m-regularity lower bound will work. So take the m delivered by Mumford's theorem 4.13 for the constants n, p, Φ. The arbitrary sheaf $\mathcal{F}$ taken above, which is not necessarily flat, then has m-regular fibers on X_t everywhere over T; with this we actually mean that $\iota_{t*}\mathcal{F}|_t$ on $\mathbb{P}^n_{k(t)}$ is m-regular, but the cohomology groups of those two sheaves coincide so we will use the concepts interchangeably.

We can also take m big enough to work for Serre vanishing of $\mathcal{E}$, meaning $R^i\pi_*\mathcal{E}(z) = 0$ for all $i \geq 1$, $z \geq m$. Trivially, $\mathcal{E}$ also has m-regular fibers $\mathcal{E}_T|_t$ on X_t everywhere over T, at least if we impose $m \geq n$. By Lemma 4.11, we also note that $g : \mathcal{G} \hookrightarrow \mathcal{E}_T$, the kernel of q, has m-regular fibers everywhere over T.

$$\begin{tikzcd}
\mathcal{G}|_t \arrow[r, hook] \arrow[dr, rightsquigarrow] & \mathcal{E}_T|_t \arrow[r, two heads] \arrow[d, rightsquigarrow] & \mathcal{F}|_t \arrow[dl, rightsquigarrow] & \mathcal{O}_{X_s}^{\oplus p} \arrow[d, two heads] & & \mathcal{G} \arrow[r, hook, "\kappa"] \arrow[dr, rightsquigarrow] & \mathcal{E}_T \arrow[r, two heads, "q"] \arrow[d, rightsquigarrow] & \mathcal{F} \arrow[dl, rightsquigarrow] \\
& X_t \arrow[r, hook, "\iota_t"] \arrow[d, "\pi_t"] \arrow[drr, dotted] & \mathbb{P}^n_{k(t)} \arrow[dl] & \mathcal{E}|_s \arrow[d, rightsquigarrow] & & & X_T \arrow[d, "\pi_T"] \arrow[dr, dotted] & \mathcal{E} \arrow[d, rightsquigarrow] \\
& Spec\, k(t) \arrow[drr, dotted, "\tau_t"] & & X_s \arrow[r, hook, "\iota_s"] \arrow[d, "\pi_s"] & \mathbb{P}^n_{k(s)} \arrow[dl] & & T \arrow[dr, dotted, "\tau"] & X \arrow[d, "\pi"] \\
& & & Spec\, k(s) & & & & S
\end{tikzcd}$$

By Castelnuovo's lemma 4.10, we have found an $m \in \mathbb{Z}$ such that for all $t \in T$, the fibers $\mathcal{G}|_t(m)$, $\mathcal{E}_T|_t(m)$ and $\mathcal{F}|_t(m)$ are generated by global sections and have vanishing higher cohomology. An argument involving Nakayama's lemma on the neighbourhoods of each fiber shows that $\mathcal{F}$ is relatively generated by global sections, i.e. that the natural map $\pi_T^*\pi_{T*}\mathcal{F}(m) \xrightarrow{ad} \mathcal{F}(m)$ corresponding to the identity of $\pi_{T*}\mathcal{F}(m)$ through the π_T^*/π_{T*} adjointness property is surjective. The same holds for $\mathcal{G}(m)$ and $\mathcal{E}_T(m)$.

If in addition $\mathcal{F}$ is flat over T, i.e. it is a family of quotients parameterized by T, then we can use the semicontinuity theorems [Har77, III.12.9-11] to conclude

$$\forall i \geq 1, t \in T: \quad R^i\pi_{T*}\mathcal{F}(m) = 0 \qquad (\pi_{T*}\mathcal{F}(m))|_t \xrightarrow{\sim} \Gamma(X_t, \mathcal{F}|_t(m)).$$

So m is a lower bound for Serre vanishing of $\mathcal{F}$, and thus $\pi_{T*}\mathcal{F}(m)$ is a rank $\Phi(m)$ vector bundle, by 3.3.

Natural transformation into the Grassmannian functor

In the preceding step we have found an integer $m \in \mathbb{Z}$ such that for every $T \in \underline{\mathcal{S}ch}_S$ and $\langle \mathcal{F}, q \rangle \in \mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S}(T)$, $\pi_{T*}q(m) : \pi_{T*}\mathcal{E}_T(m) \twoheadrightarrow \pi_{T*}\mathcal{F}(m)$ is a rank $\Phi(m)$ quotient bundle. To make this vector bundle an element of a Grassmannian functor, the sheaf from which the surjection is coming from has to be the pullback by τ of a sheaf on S; this can by achieved by precomposing it by the base-change morphism $\varphi_\tau : \tau^*\pi_*\mathcal{E}(m) \to \pi_{T*}\mathcal{E}_T(m)$, discussed in A.4. So we define

$$\alpha_T(\langle \mathcal{F}, q \rangle) = \langle \pi_{T*}\mathcal{F}(m), \pi_{T*}q(m) \circ \varphi_\tau \rangle .$$

Now we check that α_T is natural in T, so let $f : P \to T$ be a morphism of locally noetherian S-schemes. This follows from the functoriality and naturality properties of the base-change morphism A.4, whose respective diagrams have been combined into the one below. Note that we use $\tau \circ f$ to reference the structure morphism of P over S because unfortunately the letter π is already taken.

$$\begin{tikzcd}
f^*\tau^*\pi_*\mathcal{E}(m) \arrow[d, two heads, "f^*\varphi_\tau"'] \arrow[dr, two heads, "\varphi_{\tau\circ f}"] & & \mathcal{E}_P(m) \arrow[rr, squiggly] \arrow[dr, two heads, "f_X^*q(m)"] & & X_P \arrow[r, "f_X"] \arrow[dd, "\pi_P"] & X_T \arrow[r, dotted, "\tau_X"] \arrow[dd, "\pi_T"] & X \arrow[dd, "\pi"] \\
f^*\pi_{T*}\mathcal{E}_T(m) \arrow[r, two heads, "\varphi_f"] \arrow[dr, two heads, "f^*\pi_{T*}q(m)"'] & \pi_{P*}\mathcal{E}_P(m) \arrow[dr, two heads, "\pi_{P*}f_X^*q(m)"] & & f_X^*\mathcal{F}(m) \arrow[ur, squiggly] & & & \\
& f^*\pi_{T*}\mathcal{F}(m) \arrow[r, "\varphi_f", "\sim"'] & \pi_{P*}f_X^*\mathcal{F}(m) \arrow[rr, squiggly] & & P \arrow[r, "f"] & T \arrow[r, dotted, "\tau"] & S
\end{tikzcd}$$

After meditating a few minutes on the origin of each arrow, we recognize the following facts. The upper-left triangular diagram is made up of surjective morphisms because $\mathcal{E}$ is globally generated, as we explain in the appendix A.5. The base-change $f^*\pi_{T*}\mathcal{F}(m) \to \pi_{P*}f_X^*\mathcal{F}(m)$ is an isomorphism by the special criterion without flatness A.6 that we already used in the proof of flattening stratification 3.5. It applies because $f_X^*\mathcal{F}$ is flat over P, by base extension, and m is a lower bound for the required properties of the fibers to hold.

Therefore, $f^*\pi_{T*}\mathcal{F}(m)$ and $\pi_{P*}f_X^*\mathcal{F}(m)$ are isomorphic as quotient bundles of $f^*\tau^*\pi_*\mathcal{E}(m)$, i.e. $f^*\alpha_T(\langle\mathcal{F},q\rangle) = \alpha_P(\langle f_X^*\mathcal{F}, f_X^*q)$ as equivalence classes in $\mathfrak{Grass}_S(\pi_*\mathcal{E}(m),\Phi(m))(P)$. We have thus defined a natural transformation into the relative Grassmannian functor.

$$\alpha: \quad \mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S} \longrightarrow \mathfrak{Grass}_S(\pi_*\mathcal{E}(m),\Phi(m))$$

α expresses $\mathfrak{Quot}$ as a representable subfunctor

Let $T \in \underline{\mathcal{S}ch}_S$. To describe the image of α we have to find a criterion for quotient bundles of $\tau^*\pi_*\mathcal{E}(m)$ to be obtainable as the projection of a family of quotients of $\mathcal{E}_T$ on X_T. For this characterization, we proceed as in our analysis of the projective bundle leading up to 4.8, where $\mathcal{O}(1)$ and $\iota_*\mathcal{O}_Y(1)$ played the role of $\mathcal{E}_T$ and $\mathcal{F}$, respectively. So consider the following diagram, from which it is apparent that $\mathcal{F}(m)$ is uniquely determined, up to isomorphism, as the cokernel of the composition $g(m) \circ ad \circ \pi_T^*\varphi_\tau|_{\mathcal{K}} = ad \circ \pi_T^*\varphi_\tau \circ \pi_T^*\kappa$, where $\mathcal{K} \overset{\kappa}{\hookrightarrow} \tau^*\pi_*\mathcal{E}(m)$ is the kernel of $\pi_{T*}q(m) \circ \varphi_\tau$.

$$\begin{tikzcd}
\pi_T^*\mathcal{K} \arrow[r, "\pi_T^*\kappa"] \arrow[d, two heads, "\pi_T^*\varphi_\tau|_{\mathcal{K}}"] & \tau_X^*\pi^*\pi_*\mathcal{E}(m) \arrow[r, two heads] \arrow[d, two heads, "\pi_T^*\varphi_\tau"] & \pi_T^*\pi_{T*}\mathcal{F}(m) \arrow[r, squiggly] \arrow[dd, two heads, "ad"] & X_T \arrow[r, dotted, "\tau_X"] \arrow[ddd, "\pi_T"] & X \arrow[ddd, "\pi"] & \pi^*\pi_*\mathcal{E}(m) \arrow[l, squiggly] \arrow[dd, two heads, "ad"] \\
\pi_T^*\pi_{T*}\mathcal{G}(m) \arrow[r] \arrow[d, two heads, "ad"] & \pi_T^*\pi_{T*}\mathcal{E}_T(m) \arrow[ur, two heads] \arrow[d, two heads, "ad"] & & & & \\
\mathcal{G}(m) \arrow[r, hook, "g(m)"] & \mathcal{E}_T(m) \arrow[r, two heads, "q(m)"] & \mathcal{F}(m) \arrow[uur, squiggly] & & & \mathcal{E}(m) \arrow[uul, squiggly] \\
\mathcal{K} \arrow[r, hook, "\kappa"] \arrow[d, two heads, "\varphi_\tau|_{\mathcal{K}}"] & \tau^*\pi_*\mathcal{E}(m) \arrow[r, two heads] \arrow[d, two heads, "\varphi_\tau"] & \pi_{T*}\mathcal{F}(m) \arrow[r, squiggly] & T \arrow[r, dotted, "\tau"] & S & \pi_*\mathcal{E}(m) \arrow[l, squiggly] \\
\pi_{T*}\mathcal{G}(m) \arrow[r, hook, "\pi_{T*}g(m)"] & \pi_{T*}\mathcal{E}_T(m) \arrow[ur, two heads, "\pi_{T*}q(m)"'] & & & &
\end{tikzcd}$$

So we consider the following transformation, lifting a quotient bundle on T to a coherent sheaf on X_T which lands in the family of quotients functor if and only if the quotient bundle is obtainable as the projection of a family of quotients. For every $\langle \mathcal{J}, j\rangle \in \mathfrak{Grass}_S(\pi_*\mathcal{E}(m), \Phi(m))(T)$ define the coherent quotient sheaf of $\mathcal{E}_T$ on X_T

$$\beta_T(\langle \mathcal{J}, j\rangle) = \langle coker(\tau_X^* \, ad \circ \pi_T^* \kappa) \otimes \mathcal{O}(-m), q\rangle \ ,$$

where $\mathcal{K} \overset{\kappa}{\hookrightarrow} \pi_{T*}\mathcal{E}_T(m)$ is the kernel of j and q is the quotient from $\mathcal{E}_T$ to its cokernel (after untwisting it by m). Note that the pullback $\tau_X^* \, ad$ of the adjoint map $ad : \pi^*\pi_*\mathcal{E}(m) \twoheadrightarrow \mathcal{E}(m)$ on X is precisely the composition $ad \circ \pi_T^* \varphi_\tau$ of the vertical surjections on X_T in the preceding diagram. Then, if $\langle \mathcal{J}, j\rangle$ is a projection of a family of quotients, $\beta_T(\langle \mathcal{J}, j\rangle)$ is its family of quotients because it fits rightly into the preceding diagram; i.e. β_T is a left inverse of α_T, which is thus injective.

$$\begin{tikzcd}
\pi_T^*\mathcal{K} \arrow[r, "\pi_T^*\kappa"] \arrow[dr] & \tau_X^*\pi^*\pi_*\mathcal{E}(m) \arrow[r, two heads, "\pi_T^* j"] \arrow[d, "\tau_X^* ad"] & \pi_T^*\mathcal{J} \arrow[r, rightsquigarrow] & X_T \arrow[r, dotted, "\tau_X"] \arrow[dd, "\pi_T"] & X \arrow[dd, "\pi"] & \pi^*\pi_*\mathcal{E}(m) \arrow[l, rightsquigarrow] \arrow[d, two heads, "ad"] \\
 & \mathcal{E}_T(m) \arrow[r, two heads, "q(m)"] & \beta_T(\langle \mathcal{J}, j\rangle)(m) \arrow[ur, rightsquigarrow] & & & \mathcal{E}(m) \arrow[ul, rightsquigarrow] \\
\mathcal{K} \arrow[r, hook, "\kappa"] & \tau^*\pi_*\mathcal{E}(m) \arrow[r, two heads, "j"] & \mathcal{J} \arrow[r, rightsquigarrow] & T \arrow[r, dotted, "\tau"] & S & \pi_*\mathcal{E}(m) \arrow[l, rightsquigarrow]
\end{tikzcd}$$

Therefore, a quotient bundle $\langle \mathcal{J}, j\rangle$ on T is in the image of α if and only if $\beta_T(\langle \mathcal{J}, j\rangle) \in \mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S}(T)$, i.e. if $\beta_T(\langle \mathcal{J}, j\rangle)$ is flat over T with Hilbert polynomial Φ everywhere over T.

Moreover, β_T is natural in T because for every morphism $f : P \to T$ of locally noetherian S-schemes, the pullback f_X^* commutes with taking a cokernel, as it is right exact, and commutes along the diagram $\pi_T \circ f_X = f \circ \pi_P$.

$$\begin{aligned}
f_X^*\beta_T(\langle \mathcal{J}, j\rangle) &\cong coker(f_X^*(\tau_X^* \, ad \circ \pi_T^*\kappa))(-m) \\
&\cong coker((\tau \circ f)_X^* \, ad \circ \pi_P^* f^*\kappa)(-m) \cong \beta_P(\langle f^*\mathcal{J}, f^* j\rangle)
\end{aligned}$$

So for every f, $f^*\langle \mathcal{J}, j\rangle \in \mathfrak{Grass}_S(\pi_*\mathcal{E}(m), \Phi(m))(P)$ is in the image of α_P if and only if $f_X^*\beta_T(\langle \mathcal{J}, j\rangle) \in \mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S}(P)$, i.e. if and only if $f_X^*\beta_T(\langle \mathcal{J}, j\rangle)$ is flat over P with Hilbert polynomial Φ everywhere over P.

We recognize this as the universal property of the stratum $T_\Phi \hookrightarrow T$ of T with respect to the coherent sheaf $\beta_T(\langle \mathcal{J}, j\rangle)$ on X_T. So by the flattening stratification theorem 3.5, $\mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S}$ is representable by a locally closed subscheme

$$Quot^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S} \hookrightarrow Grass_S(\pi_*\mathcal{E}(m), \Phi(m)) \ .$$

The Quot scheme is projective over S

As we have seen before in 2.4, it is enough to prove that $\mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S}$ is a proper S-scheme for the monomorphism into $Grass_S(\pi_*\mathcal{E}(m), \Phi(m))$ to be a closed embedding. Since $Grass_S(\pi_*\mathcal{E}(m), \Phi(m))$ is itself projective over S, this will prove the claim. The scheme $Quot^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S}$ is of finite type over S, so we can use the valuative criterion for properness [Har77, II.4.7].

Let R be a DVR over S, K its function field and $j : Spec\,K \hookrightarrow Spec\,R$ the open inclusion. First we note that we can restrict to an arbitrary neighbourhood of the generic point of $Spec\,R$ in S, thus assume that X is H-projective over S; secondly, with Lemma 4.5 we can actually assume $X = \mathbb{P}^n_S$. Let us check that the restriction map

$$\mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S}(Spec\,R) \to \mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S}(Spec\,K)$$

is bijective. On the affine patches $D_+(x_i)$, coherent sheaves on $\mathbb{P}^n_K$ and $\mathbb{P}^n_R$ correspond to finitely generated $\underline{K} = K[x_{0/i}, ..., x_{n/i}]$ and $\underline{R} = R[x_{0/i}, ..., x_{n/i}]$ modules respectively.

Let $\langle \mathcal{F}, q \rangle \in \mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S}(Spec\,K)$. On each affine patch $D_+(x_i)$, the condition that a coherent quotient sheaf $p : \mathcal{E}_R \twoheadrightarrow \mathcal{M}$ on $\mathbb{P}^n_R$, with $\underline{R}$-module homomorphism $p : E_R \twoheadrightarrow M$, gets pulled back to $q : \mathcal{E}_K \twoheadrightarrow \mathcal{F}$, with $\underline{K}$-module homomorphism $q : E_K \twoheadrightarrow F$, translates to F and $M \otimes_R K$ being isomorphic as quotient modules of E_K. The image of E_R in F under q is the unique $\underline{R}$-module, up to isomorphism as quotient modules of E_R, that satisfies this condition. We explained this fact in more detail when checking properness of the simple Grassmannian in 2.3.

$$\begin{tikzcd}
Spec\,K \arrow[d, hook, "j"] & \mathbb{P}^n_K|_{D_+(x_i)} \arrow[l, "\pi"'] \arrow[d, hook] \arrow[r, mapsto, squiggly] & E_K \arrow[rr, two heads, "q"] \arrow[dr, two heads, "p\otimes\underline{K}"] & & F & \underline{K}\text{-modules} \\
 & & & M \otimes_{\underline{R}} \underline{K} \arrow[ur, dashed, "\sim"'] & & \\
Spec\,R & \mathbb{P}^n_R|_{D_+(x_i)} \arrow[l, "\pi"'] \arrow[r, mapsto, squiggly] & E_R \arrow[uu, hook] \arrow[r, two heads, "p"] & M \arrow[u, hook] & & \underline{R}\text{-modules}
\end{tikzcd}$$

So we recognize the image sheaf $p : \mathcal{E}_R \twoheadrightarrow \mathcal{M}$ of the composition $\mathcal{E}_R \to j_*\mathcal{E}_K \twoheadrightarrow j_*\mathcal{F}$ as the unique sheaf on $\mathbb{P}^n_R$, up to isomorphism as quotient sheaves of $\mathcal{E}_R$, pulling back to $\langle \mathcal{F}, q \rangle$. In addition, $\mathcal{M}$ is flat over R because on the affine patches $D_+(x_i)$ its $\underline{R}$-module M is torsion-free, being an $\underline{R}$-submodule of the torsion-free $\underline{K}$-module F. The fact that a module over a PID is flat if and only if it is torsion-free is a standard commutative algebra fact. Thus, we have verified the valuative criterion. □

It's worthy of note that to merely prove representability we could have easily reduced from the start to the H-projective case $X = \mathbb{P}^n_S$, completely analogously to the relative Grassmannian 2.9. However, with this approach it is only possible to conclude properness of the Quot scheme over S and not the projectivity. This is because we would have obtained a respresenting scheme which is a gluing of closed subschemes of many different Grassmannians $Grass_S(\pi_*\mathcal{E}(m), \Phi(m))$ and for different values of m, which doesn't (directly) imply it is a subscheme of a single bigger Grassmannian. We have written down the reduction step before realizing this problem, so the details are left here as an extra for the interested reader.

Proof (Reduction to the H-projective case $X = \mathbb{P}^n_S$) Fix an open cover $\{U_i\}_i$ of S on which there are surjections $\mathcal{O}_U^{\oplus n} \twoheadrightarrow \mathcal{V}|_U$, for some n that works for every i. For each U_i define the open subfunctor $\mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S/i}$ that associates to each S-scheme T the same set $\mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S}(T)$ if T factors through U_i, and the empty set if it doesn't.
With 2.7 we get a closed embedding $\iota : \mathbb{P}_{U_i}(\mathcal{V}|_{U_i}) \hookrightarrow \mathbb{P}^n_{U_i}$. Then, for S-schemes that do factor through U_i, with 4.5 applied to ι we get the following maps of sets natural in T.

$$\mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S/i}(T) \cong \mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}|_{U_i}/\mathbb{P}(\mathcal{V}|_{U_i})/U_i}(T) \cong \mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\iota_*\mathcal{E}|_{U_i}/\mathbb{P}^n_{U_i}/U_i}(T)$$

Thus, assuming that the theorem holds in the case $X = \mathbb{P}^n_{U_i}$, we can represent each $\mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S/i}$ by the U_i-scheme $Quot^{\Phi,\mathcal{O}(1)}_{\iota_*\mathcal{E}|_{U_i}/\mathbb{P}^n_{U_i}/U_i}$ considered as an S-scheme through the inclusion $U_i \hookrightarrow S$.

Since $\mathfrak{Quot}_{\mathcal{E}/X/S}$ is a Zariski sheaf, with 1.7 we obtain that $\mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S}$ is representable by an S-scheme $Quot^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S}$ which is a gluing of projective locally noetherian S-schemes.

Since proper morphisms are preserved under gluing along open covers and base extension, the resulting representing scheme is proper and locally noetherian over S, but not necessarily projective. $\square$

Version of Altman and Kleiman

Altman and Kleiman have proved in [AK80, 2.6] that under some additional assumptions the *Quot* scheme is strongly projective over S; namely, in the case that X is strongly projective and $\mathcal{E}$ is the quotient sheaf of $\pi^*\mathcal{W}(N)$ for some $N \in \mathbb{Z}$ and a finite rank vector bundle $\mathcal{W}$ on S.

First, as we did above with Lemmata 4.5 and 4.6, they reduce to $N = 0$ and $X = \mathbb{P}_S(\mathcal{V})$ for some finite rank vector bundle $\mathcal{V}$ on S. Then, they reduce to $\mathcal{E} = \pi^*\mathcal{W}$ with Lemma 4.15 below, which is a generalization of 2.7 and works for a general projective scheme X. Afterwards, their proof goes through as presented in 4.14, but with a crucial observation: the projected sheaf $\pi_*\mathcal{E}(m) = \pi_*\pi^*\mathcal{W}(m)$ is the finite rank vector bundle $\mathcal{W} \otimes_{\mathcal{O}_S} Sym_S^m\mathcal{V}$ on S. So $\mathfrak{Quot}^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S}$ becomes a closed subfunctor of $\mathfrak{Grass}_S(\mathcal{W} \otimes_{\mathcal{O}_S} Sym_S^m\mathcal{V}, \Phi(m))$, i.e. $Quot^{\Phi,\mathcal{O}(1)}_{\mathcal{E}/X/S}$ is strongly projective.

Lemma 4.15 *For all $\Phi \in \mathbb{Q}[z]$ and any surjection of coherent sheaves $\sigma : \mathcal{E}' \twoheadrightarrow \mathcal{E}$ on X, precomposition by σ induces a closed embedding of functors* $\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S} \hookrightarrow \mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}'/X/S}$.

Proof Let $T \in \underline{Sch}_S$. Denote by $\sigma_T : \mathcal{E}'_T \twoheadrightarrow \mathcal{E}_T$ the base extension of σ by τ, which is still a surjection of coherent sheaves. Precomposition by σ_T is a natural transformation in T which respects the equivalence relation in $\mathfrak{Quot}_{\mathcal{E}/X/S}(T)$ and leaves invariant the Hilbert polynomial. It is injective because an isomorphism as quotient sheaves of $\mathcal{E}'$ is also one as quotients of $\mathcal{E}$, so we have to verify that it expresses a closed subfunctor. Let $\langle \mathcal{F}, q \rangle \in \mathfrak{Quot}_{\mathcal{E}'/X/S}(T)$, and consider an arbitrary morphism of locally noetherian S-schemes $f : P \to T$.

$$\begin{tikzcd}
f_X^* ker\, \sigma_T \arrow[r, two heads] \arrow[drr, "f_X^*\gamma"'] & ker\, f_X^*\sigma_T \arrow[r, hook] & \mathcal{E}'_P \arrow[r, two heads, "f_X^*\sigma_T"] \arrow[d, "f_X^* q"] & \mathcal{E}_P \arrow[dl, dashed] \arrow[d] \\
 & & f_X^*\mathcal{F} \arrow[r, two heads] & coker\, f_X^*\gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
ker\, \sigma_T \arrow[r, hook] \arrow[dr, "\gamma"'] & \mathcal{E}'_T \arrow[r, two heads, "\sigma_T"] \arrow[d, "q"] & \mathcal{E}_T \arrow[dl, dashed] \arrow[d] \\
 & \mathcal{F} \arrow[r, two heads] & coker\, \gamma
\end{tikzcd}$$

$f_X^* q$ factors through $f_X^*\sigma_T$, i.e. $f_X^*\langle \mathcal{F}, q \rangle \in \mathfrak{Quot}_{\mathcal{E}/X/S}(P)$, if and only if its restriction to the kernel $ker\, f_X^*\sigma_T$ is zero. By precomposing with the surjection from $f_X^* ker\, \sigma_T$, which is a result of the right exactness of the pullback, this holds if and only if $f_X^* ker\, \sigma_T \to f_X^*\mathcal{F}$ is zero. This map is precisely the pullback $f_X^*\gamma$ of the composition $\gamma : ker\, \sigma_T \hookrightarrow \mathcal{E}'_T \twoheadrightarrow \mathcal{F}$ on X_T.

Therefore, we are searching for a closed subscheme C such that for every morphism of locally noetherian S-schemes $f : P \to T$, f factors through C if and only if $f_X^*\gamma$ is zero. We've referenced without proof the general theorem asserting the existence of such a subscheme in the appendix A.3. □

Appendix A

Appendix

Lemma A.1 *There is an isomorphism between the two following contravariant functors $\mathcal{S}ch \to \mathcal{S}et$.*

The functor $\mathcal{S}ubs$ associates to each scheme T the set

$$\{X \subseteq T \mid X \text{ is a closed subscheme of } T\}$$

and to a morphism $f : P \to T$ the function $X \in \mathcal{S}ubs(T) \mapsto X \times_T P \in \mathcal{S}ubs(P)$.

The functor $\mathcal{Q}coh$ associates to each scheme T the set

$$\left\{ (\mathcal{F}, q) \;\middle|\; \begin{array}{c} \mathcal{F} \text{ is a quasi-coherent } \mathcal{O}_T\text{-module} \\ q : \mathcal{O}_T \twoheadrightarrow \mathcal{F} \end{array} \right\} \Big/ \sim$$

under the equivalence relation of isomorphism as quotient sheaves of $\mathcal{O}_T$,
and to a morphism $f : P \to T$ the function $\langle \mathcal{F}, q\rangle \mapsto \langle f^*\mathcal{F}, f^*q\rangle$.

Proof The natural transformation acts on a scheme T by the function $\langle X, \iota \mapsto \langle \iota_*\mathcal{O}_X, \iota^\#\rangle$, where ι is the closed embedding $X \hookrightarrow T$.
The inverse acts on T by the function $\langle \mathcal{F}, q\rangle \mapsto (Supp\mathcal{F}, \mathcal{F}|_{Supp\mathcal{F}})$, where $Supp\mathcal{F}$ is the schematic support of $\mathcal{F}$ with structure sheaf $\mathcal{F}|_{Supp\mathcal{F}}$.

The well definedness and bijectivity of this transformation is just a rephrasing of the famous correspondence between closed subschemes and quasi-coherent sheaves of ideals [Har77, II.5.9]. In this lemma is included also a naturality assertion about this correspondence, which we will now prove.

Take an arbitrary morphism of schemes $f : P \to T$ and a closed subscheme $\iota : X \hookrightarrow T$ with corresponding quotient sheaf $\langle \mathcal{F}, q\rangle$. We have to show that $X \times_T P$ and $Suppf^*\mathcal{F}$ represent the same closed subscheme of P. This amounts to showing that $Suppf^*\mathcal{F}$ satisfies the universal property of the fibered product illustrated below.

$$\begin{tikzcd}
 & L \arrow[dr, dashed, "\widetilde{\varphi}"] \arrow[drr, bend left, "\psi"] \arrow[ddr, bend right, "\varphi"'] & & & \\
f^*\mathcal{F} & & Supp\, f^*\mathcal{F} \arrow[r, "f'"] \arrow[d, hook, "\iota'"] & X \arrow[d, hook, "\iota"] & \mathcal{F} \\
\mathcal{O}_P \arrow[u, two heads, "f^*q"'] & & P \arrow[r, "f"] & T & \mathcal{O}_T \arrow[u, two heads, "q"']
\end{tikzcd}$$

We will use the following equivalent criterion for a morphism $g : L \to T$ to factor through the closed subscheme $\iota : X \hookrightarrow T$: the surjection of $\mathcal{O}_L$-modules $g^*q : \mathcal{O}_L = g^*\mathcal{O}_T \twoheadrightarrow g^*\mathcal{F}$ must be an isomorphism [Sta24, 01HP].

The surjection $(f \circ \iota')^*q : f^*\mathcal{F}|_{Suppf^*\mathcal{F}} \cong \iota'^*\mathcal{O}_P \twoheadrightarrow \iota'^*(f^*\mathcal{F}) \cong f^*\mathcal{F}|_{Suppf^*\mathcal{F}}$ is by definition an isomorphism, so by the criterion $f \circ \iota'$ factors through ι to a morphism $f' : Suppf^*\mathcal{F} \to X$.
For any other scheme L with commuting morphisms φ,ψ to P and X respectively, we have that $(f \circ \varphi)^*q : \varphi^*\mathcal{O}_P \twoheadrightarrow \varphi^*(f^*\mathcal{F})$ is an isomorphism because $f \circ \varphi$ factors through ι.
This implies, by the criterion applied to the subscheme $\iota' : Suppf^*\mathcal{F} \hookrightarrow P$, that φ factors through ι' to a morphism $\widetilde{\varphi}$. $\widetilde{\varphi}$ is unique because ι' is a monomorphism, so the universal property is verified. $\square$

Lemma A.2 *Let T be a scheme and $\gamma : \mathcal{K} \to \mathcal{F}$ a map of quasi-coherent sheaves on T with $\mathcal{F}$ locally free. Then there exists a closed subscheme $C \subseteq T$ such that for every morphism $f : P \to T$, f factors through C if and only if $f^*\gamma : f^*\mathcal{K} \to f^*\mathcal{F}$ is the zero map.*

$$\begin{tikzcd} f^*\mathcal{K} \arrow[d, "f^*\gamma"] & & \mathcal{K} \arrow[d, "\gamma"] \\ f^*\mathcal{F} \arrow[d, squiggly] & & \mathcal{F} \arrow[d, squiggly] \\ P \arrow[r, dashed] \arrow[rr, bend left, "f"] & C \arrow[r, hook] & T \end{tikzcd}$$

Proof Since $f^*\gamma$ is zero if and only if it is zero on its restriction to an open cover of P, the functor corresponding to the universal property we wish to satisfy is a Zariski sheaf. Therefore, by 1.8 it is enough to find C on an open cover of T. So we can reduce to the special case $T = Spec\, A$, $\mathcal{F}$ is free and $\mathcal{K}$ is the quotient of a free sheaf.

Having assumed that T is affine, $\mathcal{F}$ corresponds to a free A-module A^J and $\mathcal{K}$ corresponds to an A-module which admits a surjection $\rho : A^I \twoheadrightarrow \mathcal{K}$, for some arbitrary index sets I,J. Then the composition $\gamma \circ \rho : A^I \to A^J$ corresponds to a (potentially infinite) matrix of elements $(a_{ij})_{ij}$. Let $\mathfrak{a} \subseteq A$ be the ideal generated by the entries a_{ij}. Then we claim that the closed subscheme $C = Spec\, A/\mathfrak{a}$ satisfies the desired property.

Take any morphism $f : P \to T$, assuming without loss of generality $P = Spec\, B$ with f induced by a ringhomomorphism $\varphi : A \to B$. Then we get the following string of equivalencies:

f factors through $C \iff \mathfrak{a} \subseteq ker\varphi \iff \forall i,j : \varphi(a_{ij}) = 0 \iff f^*(\gamma \circ \rho) = (\gamma \circ \rho) \otimes_A id_B : B^I \to B^J$ is zero, as it's represented by the matrix $(\varphi(a_{ij}))_{ij} \iff f^*\gamma$ is zero, as $f^*\rho$ is a surjection. □

The following is a substantial generalization of the previous lemma, but for the proof we refer to [Nit05, 3.6] as it falls outside the scope of this paper.

Theorem A.3 *Let T be a locally noetherian scheme, X a projective T-scheme and $\gamma : \mathcal{K} \to \mathcal{F}$ a map of coherent sheaves on X with $\mathcal{F}$ flat over T. Then there exists a closed subscheme $C \subseteq T$ such that for every morphism $f : P \to T$ of locally noetherian schemes, f factors through C if and only if $f_X^*\gamma : f_X^*\mathcal{K} \to f_X^*\mathcal{F}$ is the zero map.*

$$\begin{tikzcd} f_X^*\mathcal{K} \arrow[d, "f_X^*\gamma"] & & \mathcal{K} \arrow[d, "\gamma"] \\ f_X^*\mathcal{F} \arrow[d, squiggly] & & \mathcal{F} \arrow[d, squiggly] \\ X_P \arrow[rr, "f_X"] \arrow[d] & & X_T \arrow[d] \\ P \arrow[r, dashed] \arrow[rr, bend left, "f"] & C \arrow[r, hook] & T \end{tikzcd}$$

Following is the base-change lemma without the flatness assumption, we state it here without proof. It is used multiple times throughout the paper, although in practice we will apply its versions below.

Lemma A.4 *Let T be a locally noetherian scheme, $\langle X, \pi\rangle$ a projective T-scheme, $\mathcal{O}(1)$ a relatively very ample line bundle on X and $\mathcal{F}$ a coherent sheaf on X. Then, for every morphism $f : P \to T$ of locally noetherian schemes there is a natural base-change homomorphism $\varphi : f^*\pi_*\mathcal{F} \to \pi_{P*}f_X^*\mathcal{F}$.*

$$\begin{tikzcd}
 & f_X^*\mathcal{F} \arrow[r, squiggly, "" , -|] & X_P \arrow[r, "f_X"] \arrow[d, "\pi_P"] & X & \mathcal{F} \arrow[l, squiggly, -|] \\
f^*\pi_*\mathcal{F} \arrow[r, "\varphi"] & \pi_{P*}f_X^*\mathcal{F} \arrow[r, squiggly, -|] & P \arrow[r, "f"] & T \arrow[from=u, "\pi"] & \pi_*\mathcal{F} \arrow[l, squiggly, -|]
\end{tikzcd}$$

Moreover, there is an $N \in \mathbb{N}$, depending on f, such that for all $z \geq N$, the base-change $\varphi(z) : f^*\pi_*\mathcal{F}(z) \to \pi_{P*}f_X^*\mathcal{F}(z)$ is an isomorphism.

It is important to note that the base-change morphism is functorial in P and natural in $\mathcal{F}$.

For every other morphism $g : L \to P$ of locally noetherian schemes the diagram of sheaves on the right commutes.

$$\begin{tikzcd}
g^*f^*\pi_*\mathcal{F} \arrow[d, "g^*\varphi_f"] \arrow[dr, "\varphi_{f\circ g}"] & g_X^*f_X^*\mathcal{F} \arrow[r, squiggly, -|] & X_L \arrow[d, "\pi_L"] \\
g^*\pi_{P*}f_X^*\mathcal{F} \arrow[r, "\varphi_g"] & \pi_{L*}g_X^*f_X^*\mathcal{F} \arrow[r, squiggly, -|] & L
\end{tikzcd}$$

For every other coherent sheaf $\mathcal{G}$ with a morphism $q : \mathcal{G} \to \mathcal{F}$ the diagram of sheaves on the right commutes.

$$\begin{tikzcd}
 & & f_X^*\mathcal{G} \arrow[rr, squiggly, -|] \arrow[dr, "f_X^*q"] & & X_P \arrow[dd, "\pi_P"] \\
f^*\pi_*\mathcal{G} \arrow[r, "\varphi"] \arrow[dr, "f^*\pi_{P*}q"'] & \pi_{P*}f_X^*\mathcal{G} \arrow[dr, "\pi_{P*}f_X^*q"] & & f_X^*\mathcal{F} \arrow[ur, squiggly, -|] & \\
 & f^*\pi_*\mathcal{F} \arrow[r, "\varphi"] & \pi_{P*}f_X^*\mathcal{F} \arrow[rr, squiggly, -|] & & P
\end{tikzcd}$$

Lemma A.5 *In the context of Lemma A.4 above, if $\mathcal{F}$ is globally generated then the base-change φ is surjective, for every morphism f.*

Proof This follows from the naturality property above by choosing a surjection $q : \mathcal{O}_{X_T}^{\oplus r} \twoheadrightarrow \mathcal{F}$ and then restricting the commutative diagram to each fiber of P.

Namely, for each $p \in P$, we obtain that the restriction of the base-change to the fibers is surjective everywhere. As a consequence, an argument involving Nakayama's lemma on small neighbourhoods of each p proves that the whole base-change is surjective.

$$\begin{tikzcd}
k(p)^r \arrow[r, "\sim"] \arrow[d, two heads, "f^*\pi_*q|_p"] & k(p)^r \arrow[d, two heads, "\pi_{P*}f_X^*q|_p"] \\
(f^*\pi_*\mathcal{F})|_p \arrow[r, "\varphi|_p"] & (\pi_{P*}f_X^*\mathcal{F})|_p
\end{tikzcd}$$

$$f^*\pi_*\mathcal{F} \xrightarrow{\varphi} \pi_{P*}f_X^*\mathcal{F}$$

There is a less general, but in a sense stronger, version of the base-change lemma A.4. With some additional assumptions, it guarantees a lower bound for the base-change $\varphi(z)$ to be an isomorphism indipendently of f, which is what we need in the proofs of the flattening stratification theorem 3.5 and of the main existence theorem 4.14. In these instances, we use this version of the base-change lemma instead of the one provided by the authors of [Nit05] because it is not clear to us how theirs would apply.

Lemma A.6 *In the context of Lemma A.4 above, assume additionally that $f_X^*\mathcal{F}$ is flat and there is a number $N \in \mathbb{N}$ which satisfies*

$$\forall i \geq 1, z \geq N, t \in T: \quad H^i(X_t, \mathcal{F}|_t(z)) = 0 \quad (\pi_*\mathcal{F}(z))|_t \xrightarrow{\sim} \Gamma(X_t, \mathcal{F}|_t(z))\,.$$

Then, N is a lower bound for Serre vanishing of $f_X^*\mathcal{F}$, and for all $z \geq N$ the base-change $\varphi(z) : f^*\pi_*\mathcal{F}(z) \to \pi_{P*}f_X^*\mathcal{F}(z)$ is an isomorphism.

Proof Since morphisms of schemes glue and they can be verified to be isomorphisms on an open cover, the assertion is local on both P and T.

As in the proof of Lemma 3.3, we can assume without loss of generality that T is noetherian and there is a closed embedding of T-schemes $\iota : X \hookrightarrow \mathbb{P}^n_T$, and thus $\iota_P : X_P \hookrightarrow \mathbb{P}^n_P$ by base extension. Then, just as in that proof, the assertion reduces to the H-projective case $X = \mathbb{P}^n_T$ because cohomology and higher direct images can be computed in the ambient scheme and $f_X^*\mathcal{F} = \iota_P^* f_{\mathbb{P}^n_T}^* \iota_*\mathcal{F}$. So in the following, X, X_t, X_P and X_p will denote $\mathbb{P}^n_T$, $\mathbb{P}^n_{k(t)}$, $\mathbb{P}^n_P$ and $\mathbb{P}^n_{k(p)}$ respectively, where $k(t)$ and $k(p)$ are the residue fields at $t \in T$ and $p \in P$.

For all $p \in P$, we can compute the cohomology on the fibers through the field extensions $k(f(p)) \to k(p)$, obtaining:

$$\forall i, z \in \mathbb{N} : H^i(X_p, (f_X^*\mathcal{F})|_p(z)) \cong H^i(X_{f(p)}, \mathcal{F}|_{f(p)}(z)) \otimes_{k(f(p))} k(p)\,.$$

In particular, by the assumed properties of N we get

$$\forall i \geq 1, z \geq N, p \in P : \qquad H^i(X_{k(p)}, (f_X^*\mathcal{F})|_p(z)) = 0$$

$$\Gamma(X_p, (f_X^*\mathcal{F})|_p(z)) \cong (\pi_*\mathcal{F}(z))|_{f(p)} \otimes_{k(f(p))} k(p) \cong (f^*\pi_*\mathcal{F}(z))|_p\,,$$

where the last equality holds by commutativity of the pullbacks.

With the semicontinuity theorems [Har77, III.12.9-11] applied to $f_X^*\mathcal{F}$ over P, which has been assumed to be flat, we obtain that N works as a Serre vanishing bound.

$$\forall i \geq 1, z \geq N, p \in P : \ R^i\pi_{P*}f_X^*\mathcal{F}(z) = 0 \ \ (\pi_{P*}f_X^*\mathcal{F}(z))|_p \xrightarrow{\sim} \Gamma(X_p, (f_X^*\mathcal{F})|_p(z))$$

$$\begin{tikzcd}
\Gamma(X_p, (f_X^*\mathcal{F})|_p(z)) & & f_X^*\mathcal{F} \arrow[r, squiggly, mapsto] & X_P \arrow[r, "f_X"] \arrow[dd, bend left, "\pi_P"] & X \arrow[dd, "\pi"] & \mathcal{F} \arrow[l, squiggly, mapsto] \\
(f^*\pi_*\mathcal{F}(z))|_p \arrow[u, "\wr"] \arrow[r] & (\pi_{P*}f_X^*\mathcal{F}(z))|_p \arrow[ul, "\sim"'] \arrow[rr, squiggly, mapsto] & & k(p) \arrow[d, hook] & & \\
f^*\pi_*\mathcal{F}(z) \arrow[r, "\varphi(z)"] & \pi_{P*}f_X^*\mathcal{F}(z) \arrow[rr, squiggly, mapsto] & & P \arrow[r, "f"] & T & \pi_*\mathcal{F}(z) \arrow[l, squiggly, mapsto]
\end{tikzcd}$$

So the restriction $\varphi(z)|_p : (f^*\pi_*\mathcal{F}(z))|_p \xrightarrow{\sim} (\pi_{P*}f_X^*\mathcal{F}(z))|_p$ to each residue field $k(p)$ of P is an isomorphism, and thus $\varphi(z) : f^*\pi_*\mathcal{F}(z) \to \pi_{P*}f_X^*\mathcal{F}(z)$ is a surjection for all $z \geq N$.

We can apply Lemma 3.3 to obtain that $\pi_{P*}f_X^*\mathcal{F}(z)$ is locally free for all $z \geq N$. Since $\varphi(z)$ is a surjection between coherent sheaves of same fiber dimension everywhere and with locally free target, it is actually an isomorphism. This holds because we can restrict to open subsets $U \subseteq P$ where $f^*\pi_*\mathcal{F}(z)$ is a quotient of $\mathcal{O}_U^{\oplus p}$ and $\pi_{P*}f_X^*\mathcal{F}(z)|_U \cong \mathcal{O}_U^{\oplus p}$, where p is their fiber dimension, and every surjection $\mathcal{O}_U^{\oplus p} \twoheadrightarrow \mathcal{O}_U^{\oplus p}$ is an isomorphism. $\square$

Bibliography

[AK80] Allen Altman and Steven Kleiman. *Compactifying the Picard scheme, Advances in mathematics 35 (50-112)*. Springer, 1980.

[Bej20] Dori Bejleri. Math259x - Moduli Spaces in Algebraic Geometry, 2020.

[Har77] Robin Hartshorne. *Algebraic Geometry*. Springer, 1977.

[Mum66] David Mumford. *Lectures on curves on an algebraic surface*. Princeton University Press, 1966.

[Nit05] Nitin Nitsure. *Construction of Hilbert and Quot schemes*. School of Mathematics, Tata Institute of Fundamental Research, 2005.

[Sta24] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2024.

[Vak24] Ravi Vakil. The Rising Sea: Foundations of Algebraic Geometry, draft, 2024.