Eidgenössische Technische Hochschule Zürich
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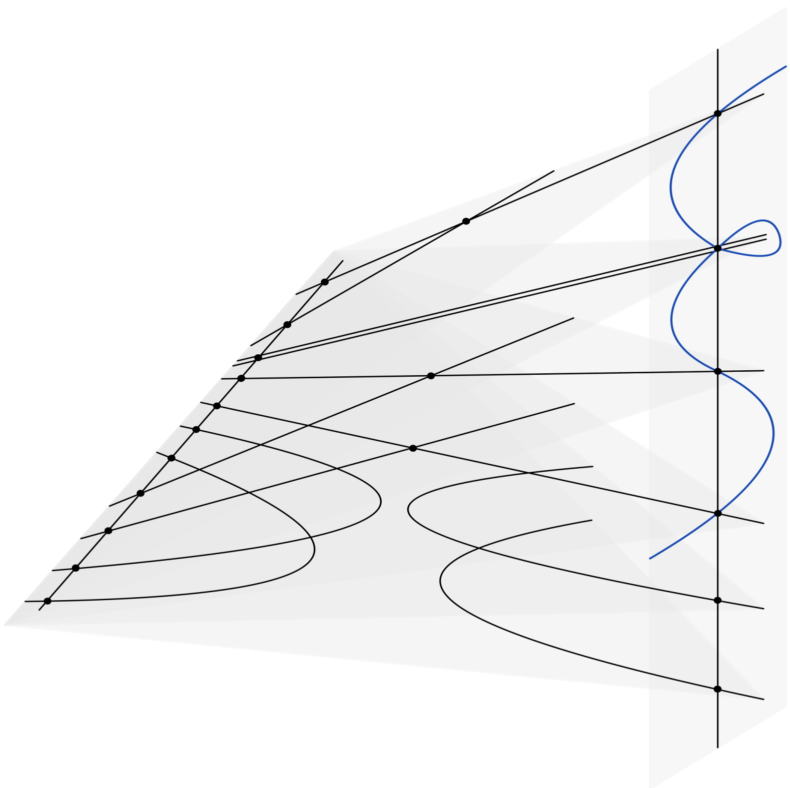
The Intermediate Jacobian of the Cubic Threefold



Master's Thesis

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If the Scatman can do it, so can you.

— Scatman John

Abstract

Intermediate Jacobians of complex projective varieties are complex tori with a polarization defined in terms of Hodge theory. We investigate their relation to algebraic cycles through Griffiths' generalized Abel–Jacobi map.

We focus on smooth cubic threefolds, studying their cohomology, their geometry of lines, and the associated Fano surface. The cubic is naturally realized as a conic bundle, which can be used to describe the intermediate Jacobian as a Prym variety of an étale double cover of curves.

Finally, we apply the results obtained to study the theta divisor, which has a unique singularity with tangent cone isomorphic to the cubic threefold itself. This allows us to deduce several important theorems, most notably that a smooth cubic threefold is not rational, a Torelli theorem for cubic threefolds, and a description of the theta divisor in terms of the Albanese embedding of the Fano surface of lines.

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Contents

Notation and conventions

Our primary references are [Har77] for general algebraic geometry, [EH16] for intersection theory, [Huy05] and [Voi09] for complex geometry, and [BL04] for complex Abelian varieties.

Varieties are integral, separated schemes of finite type over $\mathbb{C}$. Zariski-closed subsets of a scheme are interpreted as reduced subschemes. Unless stated otherwise, algebraic varieties are interpreted geometrically, meaning as sets of closed points. Singular complex projective varieties are considered as complex analytic spaces. Fiber bundles are considered in their geometric realization as spaces of closed points lying over the base.

$\Re z$, $\Im z$	Real and imaginary parts.
$\mathrm{S}^\bullet V = \bigoplus_{k\geq 0} \mathrm{S}^k V$	Symmetric algebra of a vector space.
$\mathbb{P}V, \mathbb{P}\mathcal{V}$	Projectivized space of lines in a vector space/vector bundle.
$\mathcal{O}_X(d)$	Twisted sheaf of a projective variety.
$\mathcal{S}_X = \mathcal{O}_X(-1) := \mathcal{S}_{\mathbb{P}^n}\|_X$	Tautological line bundle of a projective variety $X \subset \mathbb{P}^n$.
$\mathrm{H}_k(X, \mathbb{Z})$, $\mathrm{H}^k(X, \mathbb{Z})$	Singular (co)homology with integral coefficients.
$\mathrm{H}^k(X, \mathbb{R})$, $\mathrm{H}^k(X, \mathbb{C})$	Singular/de Rham cohomology with field coefficients.
$[Y] \in \mathrm{H}^{2n-2k}(X, \mathbb{Z})$	Fundamental cohomology class of a compact k-dimensional submanifold $Y \subseteq X$.
$\mathrm{h}^k(X) := \dim \mathrm{H}^k(X, \mathbb{Q})$	Betti numbers.
$\mathrm{e}(X)$	Topological Euler–Poincaré characteristic.
$\mathcal{A}^k_X$, $\mathcal{A}^{p,q}_X$	Sheaves of smooth, complex-valued, differential k-forms and (p, q)-forms on X.
Ω^p_X, $\overline{\Omega}^p_X$	Sheaves of holomorphic and anti-holomorphic p-forms.
$\mathrm{H}^{p,q}(X) := \mathrm{H}^q(X, \Omega^p_X)$	Dolbeault cohomology.
$\mathrm{h}^{p,q}(X) := \dim \mathrm{H}^{p,q}(X)$	Hodge numbers.
$Z_k(X)$	Group of algebraic k-cycles on an algebraic variety.
$\mathrm{CH}_k(X)$	Chow group of cycles modulo rational equivalence.
$\mathrm{CH}_k(X)_{\mathrm{alg}}, \mathrm{CH}_k(X)_{\mathrm{hom}}$	Algebraically and homologically trivial Chow groups.

Introduction

A central theme of algebraic geometry is the classification of varieties up to birational equivalence, and in particular the identification of those varieties birational to projective space, called *rational*. More generally, a variety is called *unirational* if it admits a dominant rational map from a projective space. While these notions coincide for curves and surfaces, the former is strictly stronger than the latter in higher dimensions, as we are going to see.

The question of which unirational varieties are actually rational became known as the *Lüroth problem*, following a proof by J. Lüroth in 1875 [Lü75] that a rational parametrization $\mathbb{P}^1 \dashrightarrow C$ of a curve can be reparametrized to a birational map $\mathbb{P}^1 \overset{\sim}{\dashrightarrow} C$. An analogous result for algebraic surfaces was later established by Castelnuovo with his criterion of rationality.

Many examples of unirational but non-rational varieties have since been found, most notably a series of breakthroughs all appeared around 1971-1972; we recommend [BHVK15] for a comprehensive survey. One of them is the landmark paper by Clemens and Griffiths [CG72], where they show that a smooth cubic threefold in complex projective space is not rational. The goal of this thesis is to present their main results, and to develop some of the technical tools required that are still important today.

We now give an overview of the content covered. Let $X \subseteq \mathbb{P}^4$ be a smooth cubic threefold. Its rich geometry of lines will allow us to prove that X is unirational, rationally connected, and its blow-up $\mathrm{Bl}_L X$ along a line $L \subset X$ is a bundle of plane conics over $\mathbb{P}^2$. The key invariant used by Clemens and Griffiths is the intermediate Jacobian $\mathrm{J}^3 X$, an Abelian variety whose factors not isomorphic to Jacobians of curves turn out to be a birational invariant. After constructing the smooth surface $\mathrm{F}X$ that parametrizes lines in

X, and studying its relation to the conic bundle $\mathrm{Bl}_L X \twoheadrightarrow \mathbb{P}^2$, we will obtain [CG72, §11]:

Theorem 2.35 (Clemens-Griffiths) *The intermediate Jacobian* $\mathrm{J}^3 X$ *is isomorphic as an Abelian variety to the Picard variety* $\mathrm{Pic}^0 \mathrm{F}X$*.*

Clemens and Griffiths then prove the irrationality of X and the Torelli theorem for cubic threefolds by using degeneration arguments [CG72, §13]. We will instead take the alternative route outlined in [CG72, §C], attributed to Mumford. Choosing a generic line $L \subset X$, the discriminant of the conic bundle $\mathrm{Bl}_L X \twoheadrightarrow \mathbb{P}^2$ induces an étale double cover of smooth curves $C_L \twoheadrightarrow D_L \subset \mathbb{P}^2$. This will allow us to reformulate Theorem 2.35 in terms of the so-called *Prym variety* $\mathrm{P}(C_L/D_L)$, an Abelian subvariety of $\mathrm{J}\, C_L$.

Theorem 4.5 (Mumford) *The intermediate Jacobian* $\mathrm{J}^3 X$ *is isomorphic as an Abelian variety to the Prym variety* $\mathrm{P}(C_L/D_L)$*.*

After developing Mumford's theory of Prym varieties [Mum74], the geometry of the theta divisor of $\mathrm{P}(C_L/D_L)$ will become accessible in terms of curve theory of C_L and D_L.

Theorem 4.6 *The theta divisor* Ξ *of* $\mathrm{P}(C_L/D_L)$ *has* $\mathcal{O}_{C_L}$ *as unique singular point.*

The main theorem of [CG72, §13] now follows.

Corollary 4.7 (Clemens-Griffiths) *The intermediate Jacobian* $\mathrm{J}^3 X$ *is not isomorphic as an Abelian variety to a product of Jacobians of curves. In particular,* X *cannot be rational.*

The Plücker embedding $\mathrm{F}X \hookrightarrow \mathbb{P}^9$ had been studied already in 1904 by G. Fano [Fan04]. Applying his classical results and following [Tyu70], we will prove the tangent bundle theorem. There are two other proofs of this theorem in [CG72, §12] that use the theory of residues and the Gauss mapping of $\mathrm{F}X$ respectively.

Theorem 4.11 (Clemens-Griffiths, Tyurin) *There is an isomorphism between the tangent bundle and the tautological bundle* $\mathcal{S}_{\mathrm{F}X}$ *of the Plücker embedding:* $\mathcal{T}_{\mathrm{F}X} \xrightarrow{\sim} \mathcal{S}_{\mathrm{F}X}$*.*

The isomorphism of these rank 5 vector bundles will be related to the embedding $X \subset \mathbb{P}^4$ through the projectivized differential of the Albanese map $\mathrm{F}X \hookrightarrow \mathrm{Alb}\,\mathrm{F}X$. This will imply

Theorem 4.14 *The unique singular point of* Ξ *has projectivized tangent cone isomorphic to the cubic threefold* X*.*

An immediate consequence is the Torelli theorem for cubic threefolds [CG72, §13].

Theorem 4.15 (Clemens-Griffiths) *Two smooth cubic threefolds* $X, X' \subset \mathbb{P}^4$ *are isomorphic if and only if their intermediate Jacobians* $(\mathrm{J}^3 X, \mathcal{H}_X)$ *and* $(\mathrm{J}^3 X', \mathcal{H}_{X'})$ *are isomorphic as Abelian varieties.*

Chapter 1

Basic Background

The group of algebraic cycles modulo rational equivalence is an important invariant of an algebraic variety. In the case of divisors of a projective variety over an algebraically closed field, there is a moduli interpretation in terms of line bundles which is usually well-understood. However, cycles of higher codimension often form a group of inaccessible complexity. When the ground field is $\mathbb{C}$, one can make topological considerations. In particular, topological cohomology theories come into play, with a coarser relation for the intersection theory of algebraic cycles: homological equivalence.

Let X be a complex projective variety, and consider the subgroups of algebraic cycles equivalent to 0 with respect to rational, algebraic, and homological equivalence:

$$\mathrm{Z}_k(X)_{\mathrm{rat}} \quad \subseteq \quad \mathrm{Z}_k(X)_{\mathrm{alg}} \quad \subseteq \quad \mathrm{Z}_k(X)_{\mathrm{hom}} \quad \subseteq \quad \mathrm{Z}_k(X).$$

Quotienting by $\mathrm{Z}_k(X)_{\mathrm{rat}}$, we obtain the Chow subgroups of algebraically and homologically trivial cycles modulo rational equivalence:

$$\mathrm{CH}_k(X)_{\mathrm{alg}} \subseteq \mathrm{CH}_k(X)_{\mathrm{hom}} \subseteq \mathrm{CH}_k(X).$$

The differences between these groups can be subtle, and understanding them sheds light on the algebraic and topological structure.

Recall the Abel-Jacobi map, which gives an isomorphism from divisors to line bundles

$$\Phi^0_X \,:\, \mathrm{CH}_{n-1}(X)_{\mathrm{alg}} = \mathrm{CH}_{n-1}(X)_{\mathrm{hom}} \xrightarrow{\sim} \mathrm{Pic}^0 X \cong \mathrm{H}^1(X,\mathcal{O}_X)\,/\,\mathrm{H}^1(X,\mathbb{Z})\,,$$

where we look at the Picard group as a complex torus through the isomorphism arising from the exponential sequence $\underline{\mathbb{Z}} \hookrightarrow \mathcal{O}_X \twoheadrightarrow \mathcal{O}_X^*$. Using Hodge theory, one obtains $\mathrm{H}^1(X,\mathcal{O}_X)\,/\,\mathrm{H}^1(X,\mathbb{Z}) \cong \mathrm{H}^0(X,\overline{\Omega}_X)\,/\,\mathrm{H}^1(X,\mathbb{Z}) =: \mathrm{J}^{2n-1}\,X$.

The Hodge decomposition gives a powerful tool for reducing the study of the complex structure to the linear algebra of cohomology groups. Classical examples are the Albanese and Picard varieties, which encode divisors and line bundles up to rational equivalence into the points of complex tori. We present here the generalizations $\mathrm{J}^{2k+1} X$ given by Griffiths [Gri70, **A**] of these complex tori in all odd cohomological degrees, which are well-defined in the broader category of compact Kähler manifolds. On the other hand, the Chow groups $\mathrm{CH}_k(X)_{\mathrm{alg}}$ defined purely in terms of the algebraic structure of X will be related to $\mathrm{J}^{2k+1} X$ through the generalized Abel-Jacobi map Φ^k_X. For a robust treatment we refer to [Voi09].

1.1 Intermediate Jacobians

Let X be a complex projective manifold of dimension n. There is the Hodge decomposition

$$\mathrm{H}^k(X, \underline{\mathbb{C}}) \xrightarrow{\sim} \bigoplus_{p+q=k} \mathrm{H}^{p,q}(X). \tag{1.1}$$

Fix $0 \leq k \leq n-1$ for the rest of this section, the top group $\mathrm{CH}_n(X)_{\mathrm{hom}}$ being trivial. To each homologically trivial algebraic k-cycle $[Z] \in \mathrm{CH}_k(X)_{\mathrm{hom}}$, and choosing a singular chain $W \in \mathrm{C}^{\mathrm{sin}}_{2k+1}(X, \mathbb{Z})$ with $\partial W = Z$, we can associate the functional

$$[\alpha] \in \bigoplus_{p=k+1}^{2k+1} \mathrm{H}^{p,2k+1-p}(X) \mapsto \int_W \alpha \in \mathbb{C}, \tag{1.2}$$

which is well defined because every $\beta \in \mathcal{A}^{2k}_X(X)$ with $d\beta \in \bigoplus_{r \geq k+1} \mathcal{A}^{r,2k+1-r}_X(X)$ satisfies $\int_W d\beta = \int_Z \beta = 0$ by Stokes' theorem. [1]

Thus, if we quotient by the group of all functionals induced by singular cycles through the composition

$$\mathrm{H}_{2k+1}(X, \mathbb{Z}) \xrightarrow{W \mapsto \int_W \bullet} \mathrm{H}^{2k+1}(X, \underline{\mathbb{C}})^* \xrightarrow{\rho} \textstyle\bigoplus_{p=k+1}^{2k+1} \mathrm{H}^{p,2k+1-p}(X)^*, \tag{1.3}$$

we obtain Griffiths' generalized Abel-Jacobi map [Gri70, **A**]:

$$\Phi^k_X : \mathrm{CH}_k(X)_{\mathrm{hom}} \longrightarrow \frac{\bigoplus_{p=k+1}^{2k+1} \mathrm{H}^{p,2k+1-p}(X)^*}{\rho\, \mathrm{H}_{2k+1}(X, \mathbb{Z})}.$$

[1] A general form β whose differential has terms of type (p,q) with $p < q$ does not necessarily satisfy $\int_Z \beta = 0$. This is the reason why the functional is well defined only on the left half of the Hodge diamond.

The codomain of Φ^k_X is called the *intermediate Jacobian* of X.

The intermediate Jacobian can be an interesting object to study for more general compact Kähler manifolds, to which the above definition and all the arguments until the end of this section continue to apply. The Poincaré dual maps to (1.3) are precisely the inclusion of coefficients and the projection to the right half of the Hodge diamond, also denoted ρ :

$$\mathrm{H}^{2n-2k-1}(X,\mathbb{Z}) \longrightarrow \mathrm{H}^{2n-2k-1}(X,\underline{\mathbb{C}}) \xrightarrow{\ \rho\ } \bigoplus_{p=0}^{n-k-1} \mathrm{H}^{p,2n-2k-1-p}(X). \tag{1.4}$$

Except for the annihilation of torsion, the compositions in (1.3) and (1.4) are injective because integral classes are invariant under conjugation, making the image a complete lattice of the underlying real vector space.

Definition 1.1 *The complex tori*

$$\mathrm{J}^{2k+1} X := \frac{\bigoplus_{p=k+1}^{2k+1} \mathrm{H}^{p,2k+1-p}(X)^*}{\rho\, \mathrm{H}_{2k+1}(X,\mathbb{Z})} \quad \text{and} \quad \widehat{\mathrm{J}^{2k+1}}\, X := \frac{\bigoplus_{p=0}^{k} \mathrm{H}^{p,2k+1-p}(X)}{\rho\, \mathrm{H}^{2k+1}(X,\mathbb{Z})}$$

are called the intermediate Jacobian *of X and its* dual*.* [2]

Remark 1.2 The lattice of $\bigoplus_{p=k+1}^{2k+1} \mathrm{H}^{p,2k+1-p}(X)^*$ equals to

$$\rho\, \mathrm{H}_{2k+1}(X,\mathbb{Z}) = \{\, L \mid \forall \alpha \in \mathrm{H}^{2k+1}(X,\mathbb{Z}) : 2\Re L(\overline{\rho\alpha}) \in \mathbb{Z} \}.$$

This means that $\mathrm{J}^{2k+1} X$ is the dual complex torus of $\widehat{\mathrm{J}^{2k+1}}\, X$ in the sense of Definition A.5.

An isomorphism between an intermediate Jacobian and its dual is tied to the choice of a polarization, which we will discuss in the next section. This should not be confused with the fact that, by the discussion above, there is a natural isomorphism induced by Poincaré duality

$$\mathrm{J}^{2k+1} X \cong \widehat{\mathrm{J}^{2n-2k-1}}\, X\,,$$

which will only interest us in the case $2k+1 = 2n-2k-1$, when it is precisely the isomorphism induced by the cup product polarization.

[2] A possible source of confusion is that what we defined as the dual Jacobian is often simply called Jacobian, because it is easier to define using only cohomology. In the cases of interest, the intermediate Jacobians and their duals will turn out to be isomorphic and thus identified. See Example 1.6.

Remark 1.3 A holomorphic map of compact Kähler manifolds $f : X \to Y$ induces a pullback on cohomology which is compatible with the Hodge decomposition, and thus homomorphisms of intermediate Jacobians

$$f_* := (f^*)^* : \mathrm{J}^{2k+1} X \to \mathrm{J}^{2k+1} Y, \quad f^* : \widehat{\mathrm{J}^{2k+1} Y} \to \widehat{\mathrm{J}^{2k+1} X}.$$

In other words, we have covariant and contravariant functors from compact Kähler manifolds to complex tori. If X, Y are projective, the pushforward by a proper morphism or the pullback by a flat morphism (of relative dimension 0) on the level of Chow groups commute with their counterparts on the level of intermediate Jacobians by the Abel-Jacobi map:

$$\begin{tikzcd} \mathrm{CH}_k(X)_{\mathrm{hom}} \arrow[r, "f_*"] \arrow[d, "\Phi_X^k"] & \mathrm{CH}_k(Y)_{\mathrm{hom}} \arrow[d, "\Phi_Y^k"] \\ \mathrm{J}^{2k+1} X \arrow[r, "f_*"] & \mathrm{J}^{2k+1} Y \end{tikzcd} \qquad \begin{tikzcd} \mathrm{CH}_k(X)_{\mathrm{hom}} \arrow[d, "\Phi_X^k"] & \mathrm{CH}_k(Y)_{\mathrm{hom}} \arrow[l, "f^*"'] \arrow[d, "\Phi_Y^k"] \\ \mathrm{J}^{2k+1} X & \mathrm{J}^{2k+1} Y \arrow[l, "f^*"'] \end{tikzcd}$$

Similarly to the classical case of divisors, there is a relative Abel-Jacobi map of 0 cycles.

Definition 1.4 *The Abel-Jacobi map of X with respect to the starting point $x_0 \in X$ is defined to be*

$$\Phi^0_{X,x_0} \;:\; X \xhookrightarrow{x \mapsto [x - x_0]} \mathrm{CH}_0(X)_{\mathrm{alg}} \subseteq \mathrm{CH}_0(X)_{\mathrm{hom}} \xrightarrow{\Phi^0_X} \mathrm{J}^1 X, \quad x \mapsto \left(\int_{x_0}^{x} \bullet\right).$$

1.2 Polarizations of intermediate Jacobians

Continue to consider a compact Kähler manifold X. In middle degree $\mathrm{H}^n(X, \underline{\mathbb{C}})$ we have the usual Hermitian form

$$\mathcal{H}_X \;:\; (\alpha, \beta) \mapsto 2i \int_X \alpha \wedge \overline{\beta}\,. \tag{1.5}$$

If n is odd, its imaginary part corresponds to the cup product pairing in integral cohomology $\mathrm{H}^n(X, \mathbb{Z})$ through the de Rham isomorphism and the projection ρ:

$$\int_X \alpha \smile \beta = 2\Re \int_X \rho\,\alpha \wedge \overline{\rho\,\beta} = \Im \mathcal{H}_X(\rho\,\alpha, \rho\,\beta),$$

where we use integration notation for both singular chains and differential forms while evaluating against the fundamental class. This means that $\Im\mathcal{H}_X$ restricts to an integer-valued, unimodular alternating form on $\rho\,\mathrm{H}^n(X, \mathbb{Z})$.

More generally for $2k + 1 \leq n$, with the choice of a Kähler form $\omega \in \mathrm{H}^2(X, \underline{\mathbb{C}})$, the Hodge-Riemann pairing gives on $\mathrm{H}^{2k+1}(X, \underline{\mathbb{C}})$ a Hermitian form

$$\mathcal{H}_X^{2k+1} \;:\; (\alpha, \beta) \mapsto 2i \int_X \alpha \wedge \overline{\beta} \wedge \omega^{n-2k-1}, \tag{1.6}$$

which is non-degenerate when restricted to either half of the Hodge diamond. If ω is integral, [3] the imaginary part $\Im\mathcal{H}_X^{2k+1}$ will be integer-valued on $\rho\,\mathrm{H}^{2k+1}(X,\mathbb{Z})$ [4] and the Hermitian form $\mathcal{H}_X^{2k+1}$ will induce an isogeny of complex tori $\phi_{\mathcal{H}_X^{2k+1}}$ by the correspondence (A.1):

$$\begin{tikzcd} \widehat{\mathrm{J}^{2k+1}\,X} \arrow[r, "\text{Poincaré Duality}", "\sim"'] \arrow[d, two heads, "\phi_{\mathcal{H}_{\mathbb{C}}^{2k+1}}", "\alpha \mapsto \frac{1}{2i}\mathcal{H}_X^{2k+1}(\bullet,\overline{\alpha})"'] & \mathrm{J}^{2n-2k-1}\,X \\ \mathrm{J}^{2k+1}\,X \arrow[r, "\text{Poincaré Duality}", "\sim"'] & \widehat{\mathrm{J}^{2n-2k-1}}\,X. \end{tikzcd} \tag{1.7}$$

The dualizing map $\phi_{\mathcal{H}_X^{2k+1}}$ is an isomorphism if and only if $\Im\mathcal{H}_X^{2k+1}$ is unimodular. Summarizing, if the chosen Kähler form is integral, the pair $(\widehat{\mathrm{J}^{2k+1}\,X},\mathcal{H}_X^{2k+1})$ is a polarized complex torus in the sense of Definition A.1.

Polarized complex tori whose Hermitian form is positive-definite are called *Abelian varieties* and have particularly nice properties: they are precisely the complex tori which admit an embedding in projective space; see Theorem A.2. It is thus of interest to look at the definiteness of $\mathcal{H}_X^{2k+1}$ on our intermediate Jacobians, which is described by the Hodge-Riemann bilinear relations [Huy05, 3.3.15]. Namely, for any primitive form $0 \neq \alpha \in \mathrm{H}^{p-l,q-l}(X)_{\mathrm{p}}$ in the Lefschetz decomposition [Huy05, 3.3.13], one obtains :

$$\mathrm{H}^{2k+1}(X,\underline{\mathbb{C}}) \cong \bigoplus_{p+q=2k+1}\ \bigoplus_{l=0}^{\max(p,q)} \mathrm{L}^l\,\mathrm{H}^{p-l,q-l}(X)_{\mathrm{p}}, \quad (-1)^{\frac{(2k+1)^2-1}{2}+q+l}\cdot 2i\int_X \mathrm{L}^l\,\alpha\wedge\overline{\mathrm{L}^l\,\alpha}\wedge\omega^{n-2k-1} > 0.$$

Lemma 1.5 *The Hermitian form* $(-1)^{\frac{(p+q)^2-1}{2}+q+l}\cdot\mathcal{H}_X^{2k+1}$ *is positive-definite when restricted to the summand* $\mathrm{L}^l\,\mathrm{H}^{p-l,q-l}(X)_{\mathrm{p}}$ *of the Lefschetz decomposition of* $\mathrm{H}^{2k+1}(X,\underline{\mathbb{C}})$.

Thus, if the Hodge diamond of X has a special enough pattern of trivial groups, either $\mathcal{H}_X^{2k+1}$ or $-\mathcal{H}_X^{2k+1}$ will be a positive-definite Hermitian form on either half of the Hodge diamond. If moreover $\pm\Im\mathcal{H}_X^{2k+1}$ is integer-valued as explained above, one will obtain a positive-definite polarization.

Example 1.6 The intermediate Jacobians of degree 1 of a compact Kähler manifold X are precisely the Albanese and Picard varieties [Huy05, 3.3]:

$$\mathrm{J}^1\,X = \mathrm{H}^{1,0}(X)^*/\rho\,\mathrm{H}_1(X,\mathbb{Z}) = \mathrm{Alb}\,X, \quad \widehat{\mathrm{J}^1\,X} = \mathrm{H}^{0,1}(X)/\rho\,\mathrm{H}^1(X,\mathbb{Z}) = \mathrm{Pic}^0\,X.$$

[3] An integral Kähler form exists precisely when X is projective, by the Kodaira embedding theorem.

[4] Only in odd, middle degree $2k+1=n$ the alternating form $\Im\mathcal{H}_X^{2k+1}$ is guaranteed to be unimodular on $\rho\,\mathrm{H}^{2k+1}(X,\mathbb{Z})$. In general, the Kähler form ω cannot be chosen so that $\Im\mathcal{H}_X^{2k+1}$ is unimodular because the Lefschetz map $\mathrm{L}^{n-k}:\mathrm{H}^k(X,\mathbb{Z})\hookrightarrow\mathrm{H}^{2n-k}(X,\mathbb{Z})$ is often not surjective, as we are about to see for projective hypersurfaces of degree >1.

If X is projective, the Abel-Jacobi maps become

$$\mathrm{CH}_0(X)_{\mathrm{alg}} \hookrightarrow \mathrm{CH}_0(X)_{\mathrm{hom}} \to \mathrm{J}^1\, X, \qquad \mathrm{CH}_{n-1}(X)_{\mathrm{alg}} \hookrightarrow \mathrm{CH}_{n-1}(X)_{\mathrm{hom}} \xrightarrow{\sim} \mathrm{J}^{2n-1}\, X \cong \widehat{\mathrm{J}^1\, X}. \tag{1.8}$$

If in addition an integral Kähler form is chosen, the Hermitian form $-\mathcal{H}^1_X$ becomes a positive-definite polarization on $\widehat{\mathrm{J}^1\, X}$, the indices of Lemma 1.5 being $q = 1, k = p = l = 0$, making $\widehat{\mathrm{J}^1\, X}$ and its dual an Abelian variety. In particular, this holds for a smooth projective curve C, where the polarization $-\mathcal{H}_C$ is moreover principal and the dualizing map $\phi_{-\mathcal{H}_C} : \widehat{\mathrm{J}\, C} \xrightarrow{\sim} \mathrm{J}\, C$ is precisely Poincaré duality.

Lemma 1.7 *Let $f : X \to Y$ be a holomorphic map of complex projective manifolds of dimension n with finite generic fiber of length $d \geq 1$, and ω an integral Kähler form on Y. Then, with respect to the polarizations* (1.6) *defined in terms of ω on Y and $f^*\omega$ on X, the pullback f^* induces a morphism*

$$(\widehat{\mathrm{J}^{2k+1}\, Y}, d \cdot \mathcal{H}^{2k+1}_Y) \to (\widehat{\mathrm{J}^{2k+1}\, X}, \mathcal{H}^{2k+1}_X) \ \text{ of polarized complex tori, for all } 2k+1 \leq n.$$

Proof The assumption on the generic fiber implies that f_* is multiplication by d in top integral homology, so for all $\alpha, \beta \in \mathrm{H}^{2k+1}(X, \mathbb{Z})$ it satisfies

$$\int_X f^*\alpha \smile f^*\beta \smile f^*\omega^{n-2k-1} = \int_{f_*\{X\}} \alpha \smile \beta \smile \omega^{n-2k-1} = d \int_Y \alpha \smile \beta \smile \omega^{n-2k-1}$$

by the naturality of the cap product. Then we conclude

$$\Im\mathcal{H}_X(f^*\rho\,\alpha, f^*\rho\,\beta) = \Im\mathcal{H}_X(\rho\, f^*\alpha, \rho\, f^*\beta) = d\, \Im\mathcal{H}^{2k+1}_Y(\rho\,\alpha, \rho\,\beta),$$

and since $\mathcal{H}^{2k+1}_Y$ is uniquely determined by the values of its imaginary part on the complete lattice $\rho\,\mathrm{H}^{2k+1}(Y, \mathbb{Z})$, we have verified that $\mathcal{H}^{2k+1}_X$ pulls back to $d\mathcal{H}^{2k+1}_Y$ on $\rho\,\mathrm{H}^{2k+1}(Y, \mathbb{C})$.□

1.3 Blowing up threefolds

Consider in this section a complex projective threefold X. The intermediate Jacobian $\widehat{\mathrm{J}^3\, X}$ of every such threefold has the principal polarization $\mathcal{H}_X$ described in (1.5). Armed with Hironaka's resolution of singularities, we know that to study how invariants change under rational maps it is enough to study how they change under blow-ups. Since blow-ups are degree 1 maps of same-dimensional manifolds, by Lemma 1.7 they induce embeddings of the middle degree polarized intermediate Jacobians. In the case of threefolds, the only blow-ups that will concern us are along points and smooth curves. [5]

Lemma 1.8 *For every point $p \in X$, the blow-up $\mathrm{Bl}_p X \twoheadrightarrow X$ induces an isomorphism*

$$(\widehat{\mathrm{J}^3 X}, \mathcal{H}_X) \xrightarrow{\sim} (\widehat{\mathrm{J}^3 \mathrm{Bl}_p X}, \mathcal{H}_{\mathrm{Bl}_p X}) \text{ \textit{of polarized complex tori.}}$$

Proof This is a simpler version of the proof of Lemma 1.9, where one would obtain $\mathrm{H}^3(X, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^3(\mathrm{Bl}_p X, \mathbb{Z})$ by substituting $C = \{p\}$. $\square$

Lemma 1.9 *For every smooth curve $C \subset X$, the blow-up $\mathrm{Bl}_C X \twoheadrightarrow X$ induces an isomorphism*

$$(\widehat{\mathrm{J}^3 X}, \mathcal{H}_X) \times (\widehat{\mathrm{J}^1 C}, -\mathcal{H}_C) \xrightarrow{\sim} (\widehat{\mathrm{J}^3 \mathrm{Bl}_C X}, \mathcal{H}_{\mathrm{Bl}_C X}) \text{ \textit{of polarized complex tori.}}$$

Proof Consider the exceptional divisor $E \subset \mathrm{Bl}_C X$. The blow-up morphism $\varphi : \mathrm{Bl}_C X \twoheadrightarrow X$ restricts to an isomorphism $\widetilde{U} \xrightarrow{\sim} U$ between the open sets $\widetilde{U} = \mathrm{Bl}_C X \setminus E$, $U = X \setminus C$. The long exact sequence of the pairs $(\mathrm{Bl}_C X, \widetilde{U})$ and (X, U) in singular cohomology with integral coefficients are connected by the pullback φ^*, and we will describe relative cohomology groups using the normal bundles of the submnanifolds $C \subset X$ and $E \subset \mathrm{Bl}_C X$:

$$\begin{tikzcd}
\mathrm{H}^2(U) \arrow[r, "\delta"] \arrow[d, "\wr", "\varphi^*"'] & \mathrm{H}^3(X,U) \arrow[r] \arrow[d] \arrow[ddl, "\sim"] & \mathrm{H}^3(X) \arrow[r] \arrow[d] & \mathrm{H}^3(U) \arrow[d, "\wr", "\varphi^*"'] \\
\mathrm{H}^2(\widetilde{U}) \arrow[r, "\delta"] & \mathrm{H}^3(\mathrm{Bl}_C X, \widetilde{U}) \arrow[r] \arrow[ddl, "\sim"] & \mathrm{H}^3(\mathrm{Bl}_C X) \arrow[r] & \mathrm{H}^3(\widetilde{U}) \\
\mathrm{H}^3_{\mathrm{cpt}}(\mathcal{N}_{Z/X}) & & \begin{smallmatrix}\iota : \mathcal{N}_{E/\,\mathrm{Bl}_C X} \hookrightarrow \mathrm{Bl}_C X \\ \text{tubular neighbourhood embedding}\end{smallmatrix} & \\
\mathrm{H}^3_{\mathrm{cpt}}(\mathcal{N}_{E/\,\mathrm{Bl}_C X}) \arrow[uurr, "\iota_*"] & & &
\end{tikzcd} \tag{1.9}$$

By the tubular neighbourhood theorem, there exists an open embedding $\iota : \mathcal{N}_{E/\,\mathrm{Bl}_C X} \hookrightarrow \mathrm{Bl}_C X$ such that the zero section is sent to $E \subset \mathrm{Bl}_C X$. Singular cocycles on $\mathcal{N}_{E/\,\mathrm{Bl}_C X}$ annihilating chains in $\mathcal{N}_{E \setminus \mathrm{Bl}_C X} \setminus E$ can be represented by compactly supported ones, inducing

[5] Since $\mathcal{H}_X$ induces the Poincaré duality isomorphism $\widehat{\mathrm{J}^3 X} \cong \mathrm{J}^3 X$, the results below apply just as well to the induced polarization on the complex torus $\mathrm{J}^3 X$, with analogous proofs in homology instead of cohomology.

an isomorphism which commutes with the pushforward ι_*:

$$
\begin{tikzcd}
\mathrm{H}^1(E) \arrow[r, shift left, "\pi^*\smile\Phi", "\text{Thom isomorphism}"'{above=10pt}] & \mathrm{H}^3_{\mathrm{cpt}}(\mathcal{N}_{E/\operatorname{Bl}_C X}) \arrow[l, shift left, "\sim", "\pi_*"', "\text{fiber evaluation}"'{below=10pt}] \arrow[drrr, "\iota_*"] & \mathrm{H}^3(\mathcal{N}_{E/\operatorname{Bl}_C X}, \mathcal{N}_{E/\operatorname{Bl}_C X} \setminus 0) \arrow[l, "\sim"'] & \mathrm{H}^3(\operatorname{Bl}_C X, \widetilde{U}) \arrow[l, "\text{excision}"', "\sim"] \arrow[d] \\
& & & \mathrm{H}^3(\operatorname{Bl}_C X)\,.
\end{tikzcd}
$$

The same holds for the pair (X, U), but since $\mathcal{N}_{Z/X}$ is of real rank 4 over a 2-dimensional base, we have $\mathrm{H}^3_{\mathrm{cpt}}(\mathcal{N}_{Z/X}) = 0$. This implies that $\delta : \mathrm{H}^2(\widetilde{U}) \to \mathrm{H}^3(\operatorname{Bl}_C X, \widetilde{U})$ is 0, meaning that $\mathrm{H}^3(\operatorname{Bl}_C X, \widetilde{U}) \to \mathrm{H}^3(\operatorname{Bl}_C X)$ and thus ι_* are injective.

Since φ is a degree 1 map of compact manifolds, also $\varphi^* : \mathrm{H}^3(X) \to \mathrm{H}^3(\operatorname{Bl}_C X)$ is injective. By exactness of (1.9) at $\operatorname{Bl}_C X$, we obtain an isomorphism

$$\mathrm{H}^3(X) \oplus \mathrm{H}^1(E) \xrightarrow{\sim} \mathrm{H}^3(\operatorname{Bl}_C X), \quad (\omega, \alpha) \mapsto (\varphi^*\omega + \iota_*(\pi^*\alpha \smile \Phi))\,,$$

where Φ is the Thom class of the normal bundle $\pi : \mathcal{N}_{E/\operatorname{Bl}_C X} \twoheadrightarrow E$.

The restriction $\varphi : E = \mathbb{P}(\mathcal{N}_{Z/X}) \twoheadrightarrow C$ is a $\mathbb{P}^1$-bundle, and the Euler class e of the tautological bundle $\mathcal{N}_{E/\operatorname{Bl}_C X}$ restricts to a negative generator of $\mathrm{H}^2(E|_p)$ on each fiber. Thus, the Gysin exact sequence gives isomorphisms

$$\cdots \longrightarrow \mathrm{H}^1(C) \xrightarrow[\sim]{\varphi^*} \mathrm{H}^1(E) \longrightarrow \cdots \longrightarrow \mathrm{H}^4(E) \xrightarrow[\sim]{\varphi_*} \mathrm{H}^2(C) \longrightarrow \cdots$$

where φ_* is the evaluation along the fiber with inverse $\varphi^* \smile (-\mathrm{e})$, meaning that it satisfies the projection formula

$$\int_E \varphi^*(\alpha) \smile (-\mathrm{e}) = \int_C \varphi_*(\varphi^*(\alpha) \smile (-\mathrm{e})) = \int_C \alpha, \quad \text{for all } \alpha \in \mathrm{H}^2(C).$$

Passing now to complex coefficients and projecting to the right half of the Hodge diamond of the complex manifolds $E \hookrightarrow \operatorname{Bl}_C X \twoheadrightarrow X$ we obtain the following commutative diagram. [6]

$$
\begin{tikzcd}
\mathrm{H}^3(X,\mathbb{Z}) \oplus \mathrm{H}^1(C,\mathbb{Z}) \arrow[r, "\varphi^* \oplus \iota_*(\pi^*\varphi^* \smile \Phi)", "\sim"'] \arrow[d, two heads] & \mathrm{H}^3(\operatorname{Bl}_C X,\mathbb{Z}) \arrow[d, two heads] \\
\rho\,\mathrm{H}^3(X,\mathbb{Z}) \oplus \rho\,\mathrm{H}^1(C,\mathbb{Z}) \arrow[r, "\sim"] \arrow[d, hook] & \rho\,\mathrm{H}^3(\operatorname{Bl}_C X,\mathbb{Z}) \arrow[d, hook] \\
\mathrm{H}^{1,2}(X) \oplus \mathrm{H}^{0,3}(X) \oplus \mathrm{H}^{0,1}(C) \arrow[r, "\sim"] \arrow[d, two heads] & \mathrm{H}^{1,2}(\operatorname{Bl}_C X) \oplus \mathrm{H}^{0,3}(\operatorname{Bl}_C X) \arrow[d, two heads] \\
\widehat{\mathrm{J}^3 X} \oplus \widehat{\mathrm{J}^1 C} \arrow[r, "\sim"] & \widehat{\mathrm{J}^3 \operatorname{Bl}_C X}
\end{tikzcd}
\tag{1.10}
$$

[6] Recall that $\rho\,\mathrm{H}^3(X,\mathbb{Z})$ is a complete lattice of the $\mathbb{R}$-vector space $\mathrm{H}^{1,2}(X) \oplus \mathrm{H}^{0,3}(X)$.

We now have to check that the Hermitian form on $\mathrm{H}^3(\mathrm{Bl}_C X, \mathbb{C})$ is pulled back to the ones on $\mathrm{H}^3(X, \mathbb{C})$ and $\mathrm{H}^1(C, \mathbb{C})$. Since φ is a degree 1 map of compact manifolds, it pulls back the cup product pairing on $\mathrm{Bl}_C X$ to the one on X, so $(\varphi^*)^* \mathcal{H}_{\mathrm{Bl}_C X} = \mathcal{H}_X$ is verified.

Recall that the Thom class Φ is the fundamental class of the zero section of the vector bundle $\pi : \mathcal{N}_{E/\mathrm{Bl}_C X} \twoheadrightarrow E$. We will need the self-intersection formula

$$\Phi^2 = \pi^* \mathrm{e} \smile \Phi \in \mathrm{H}^4(\mathcal{N}_{E/\mathrm{Bl}_C X}, \mathbb{Z}).$$

Take now $\alpha, \beta \in \mathrm{H}^1(C, \mathbb{Z})$ and compute

$$\begin{aligned}\int_{\mathrm{Bl}_C X} \iota_*(\pi^*\varphi^*\alpha \smile \Phi \smile \pi^*\varphi^*\alpha \smile \Phi) &= \int_{\mathcal{N}_{E/\mathrm{Bl}_C X}} \pi^*\varphi^*\alpha \smile \Phi \smile \pi^*\varphi^*\beta \smile \Phi \\ &= \int_{\mathcal{N}_{E/\mathrm{Bl}_C X}} \pi^*(\varphi^*(\alpha \smile \beta) \smile \mathrm{e} \smile \Phi \\ &= \int_E \varphi^*(\alpha \smile \beta) \smile \mathrm{e} \\ &= -\int_C \alpha \smile \beta\,.\end{aligned}$$

Then we conclude

$$\Im \mathcal{H}_{\mathrm{Bl}_C X}(\rho\ \iota_*(\pi^*\varphi^*\alpha \smile \Phi), \rho\ \iota_*(\pi^*\varphi^*\beta \smile \Phi)) = -\Im \mathcal{H}_C(\rho\ \alpha, \rho\ \beta)\,,$$

and since $\mathcal{H}_C$ is uniquely determined by the values of its imaginary part on the complete lattice $\rho\,\mathrm{H}^1(C, \mathbb{Z})$, we have verified that $\mathcal{H}_{\mathrm{Bl}_C X}$ pulls back to $-\mathcal{H}_C$ on $\mathrm{H}^{0,1}(C)$. □

We start leaving implicit in the notation the polarizations of each intermediate Jacobian and indentifying $\mathrm{J}^3 X \cong \widehat{\mathrm{J}^3 X}$ and $\mathrm{J}\,C \cong \widehat{\mathrm{J}\,C}$ under Poincaré duality. For Jacobians of curves, the implied polarization is $-\mathcal{H}$, whereas for threefolds it will be $\mathcal{H}$, because these are the ones that are positive-definite.

Corollary 1.10 *If $X \overset{\sim}{\dashrightarrow} X'$ is a birational map between complex projective threefolds, then there exist finitely many smooth, projective curves $(C_i)_i$, $(C'_j)_j$ and embeddings*

$$\mathrm{J}^3 X \times \prod_i \mathrm{J}\,C_i \hookleftarrow \mathrm{J}^3 X', \quad \mathrm{J}^3 X \hookrightarrow \mathrm{J}^3 X' \times \prod_j \mathrm{J}\,C'_j \quad \textit{of polarized complex tori.}$$

Proof Call $f : X \overset{\sim}{\dashrightarrow} X'$ the birational map and g its inverse. We build the "Hironaka house" that resolves the singularities of f and g:

$$\begin{tikzcd} \widetilde{X} \arrow[d, "p"'] \arrow[drr, "\tilde{f}" near start] & & \widetilde{X'} \arrow[d, "p'"] \arrow[dll, "\tilde{g}"' near start] \\ X \arrow[rr, dashed, shift left, "f"] & & X' \arrow[ll, dashed, shift left, "g"] \end{tikzcd}\,,$$

where $\widetilde{X}$ and $\widetilde{X'}$ are obtained from a sequence of blowing-ups p and p' of X and X', respectively, such that f lifts to a birational morphism $\widetilde{f} = f \circ p$ and g lifts to a birational morphism $\widetilde{g} = g \circ p'$.

All the maps above are of degree 1, so taking intermediate Jacobians and applying repeatedly Lemma 1.9 to the blow-ups that compose p, p', yields curves $(C_i)_i$, $(C'_j)_j$ and the following embeddings of polarized complex tori:

$$\begin{tikzcd}
\mathrm{J}^3 X \times \prod_i \mathrm{J}\, C_i \arrow[r, "\sim"] & \mathrm{J}^3 \widetilde{X} & & \mathrm{J}^3 \widetilde{X'} & \mathrm{J}^3 X' \times \prod_j \mathrm{J}\, C'_j \arrow[l, "\sim"'] \\
& \mathrm{J}^3 X \arrow[u, hook, "p^*"] \arrow[urr, hook, "\widetilde{g}^*"] & & \mathrm{J}^3 X' . \arrow[u, hook, "p'^*"'] \arrow[ull, hook, "\widetilde{f}^*"'] &
\end{tikzcd}$$

□

Because of Corollary 1.10, one could say that the intermediate Jacobian $\mathrm{J}^3 X$ of a threefold X, modulo its embeddability in products of Jacobians, is a birational invariant. This can be made precise in the following way. Consider the set $\mathfrak{G}$ of isomorphism classes of principally polarized complex tori, further quotiented by the equivalence relation:

$$A \sim B :\Longleftrightarrow \begin{array}{c} \text{there exist smooth, projective curves } (C_i)_i, (C'_j)_j \text{ and embeddings} \\ A \hookrightarrow B \times \prod_j \mathrm{J}\, C'_j,\ B \hookrightarrow A \times \prod_i \mathrm{J}\, C_i \text{ of polarized complex tori.} \end{array}$$

The cartesian product between polarized complex tori gives a well-defined, associative and commutative binary operation on $\mathfrak{G}$, making it into a commutative monoid. The neutral element in $\mathfrak{G}$ is the class of all principally polarized complex tori embeddable into a product of Jacobians.

The equivalence class of $\mathrm{J}^3 X$ in $\mathfrak{G}$, which we will denote $\mathfrak{Jac}^3 X$, is often called the *Griffiths component* of the threefold X. Its importance is explained by the following paraphrasing of Corollary 1.10.

Corollary 1.11 (Clemens-Griffiths) *The Griffiths component $\mathfrak{Jac}^3 X \in \mathfrak{G}$ is a birational invariant of X.*

Since Jacobians of curves with the implied polarization $-\mathcal{H}$ are Abelian varieties, it makes sense to talk about the submonoid $\mathfrak{A} \subset \mathfrak{G}$ of classes containing only Abelian varieties. The structure of this submonoid is easily described using the canonical decomposition of Abelian varieties into indecomposables [BL04, 4.7]. [7] This is stated more conveniently in terms of the associated theta divisors of an Abelian variety; see Definition A.8.

Theorem 1.12 *Let (A, Θ) be a principally polarized Abelian variety with its theta divisor, and $\Theta = \sum_i n_i \Theta_i$ the decomposition in irreducible components. Then, each $n_i = 1$ and there exist indecomposable, principally polarized Abelian varieties $(A_i, \Theta_i')_i$, uniquely determined up to isomorphism and permutation, such that $(A, \Theta) \cong \prod_i (A_i, \Theta_i')$ as Abelian varieties, and Θ_i is the pullback divisor of Θ_i' under the projection $\prod_j A_j \twoheadrightarrow A_i$.*

As a consequence, two principally polarized Abelian varieties A, B are equivalent in $\mathfrak{G}$ if and only if their indecomposable factors not isomorphic to Jacobians of curves coincide. Thus, we recognize $\mathfrak{A}$ as the commutative monoid freely generated by indecomposable principally polarized Abelian varieties which are not Jacobians.

1.4 Smooth hypersurfaces

We now focus on a particular class of projective manifolds. Let us fix for the rest of this section a smooth, projective hypersurface $X \subseteq \mathbb{P} := \mathbb{P}^{n+1}$ of degree d. The Lefschetz Hyperplane theorem [Voi09, 1.1] states that the pullback maps $\iota^* : \mathrm{H}^k(\mathbb{P}, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^k(X, \mathbb{Z})$ are isomorphisms for all $k < n$. Since $\mathrm{H}^k(\mathbb{P}, \mathbb{Z}) = 0$ in odd $k \neq n$, we only consider even indices. By the naturality of the cup product, the following diagram commutes:

$$\begin{tikzcd}
{[H]^{2k} \cdot \mathbb{Z} \otimes [H]^{2n-2k} \cdot \mathbb{Z}} \arrow[r, equal] & \mathrm{H}^{2k}(\mathbb{P}, \mathbb{Z}) \otimes \mathrm{H}^{2n-2k}(\mathbb{P}, \mathbb{Z}) \arrow[r, "\smile"] \arrow[d, "\iota^* \otimes \iota^*"] & \mathrm{H}^{2n}(\mathbb{P}, \mathbb{Z}) \arrow[r, equal] \arrow[d, "{[H]^n \mapsto d \cdot [*]}"] & {[H]^n \cdot \mathbb{Z}} \\
 & \mathrm{H}^{2k}(X, \mathbb{Z}) \otimes \mathrm{H}^{2n-2k}(X, \mathbb{Z}) \arrow[r, "\smile"] & \mathrm{H}^{2n}(X, \mathbb{Z}) \arrow[r, equal] & {[*] \cdot \mathbb{Z},}
\end{tikzcd}$$

where $H \subset \mathbb{P}$ is any codimension 1 hyperplane not containing X.

Lemma 1.13 *For $[H]$ the pullback of a hyperplane class in $\mathrm{H}^2(\mathbb{P}, \mathbb{Z})$, the cohomology of the smooth, degree d, n-dimensional hypersurface $X \subset \mathbb{P}$ is*

[7] Recall that a polarized complex torus is called *indecomposable* if it is not isomorphic, as a polarized complex torus, to a non-trivial product of polarized complex tori. Maximal decompositions into indecomposables always exist, but only for principally polarized Abelian varieties there is a canonical one.

$$\mathrm{H}^{2k}(X,\mathbb{Z}) = \begin{cases} \frac{1}{d}[H]^k \cdot \mathbb{Z} & \textit{for } 2k > n, \\ [H]^k \cdot \mathbb{Z} & \textit{for } 2k < n, \end{cases}$$

$$\mathrm{H}^{k}(X,\mathbb{Z}) = 0 \quad \textit{for odd } k \neq n.$$

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
\iddots & \vdots & & \vdots & & \vdots & & \vdots & \ddots \\
* & * & & * & & * & & * & * \\
\ddots & \vdots & & \vdots & & \vdots & & \vdots & \iddots \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Proof For all $2k < n$, $\mathrm{H}^{2k}(X,\mathbb{Z})$ is free of rank one because ι^* is an isomorphism. On the other hand, since the cup product induces a perfect pairing, this forces $\mathrm{H}^{2n-2k}(X,\mathbb{Z})$ to be free of rank one as well as to vanish in odd dimensions. Then, we recognize that $\iota^*([H]^{2n-2k})$ is d times a generator of $\mathrm{H}^{2n-2k}(X,\mathbb{Z})$. $\square$

The two extremes in middle cohomology are computed from the canonical bundle, known to be $\omega_X \cong \mathcal{O}_X(d-n-2)$ for degree d hypersurfaces [Huy05, 2.4.9], giving us

$$\mathrm{h}^{0,n}(X) \cong \mathrm{h}^{n,0}(X) = \dim \mathrm{H}^0(X, \mathcal{O}_X(d-n-2)) = \begin{cases} 0 & d \leq n+1 \\ \binom{d-1}{d-n-2} & d > n+1. \end{cases}$$

The total cohomology in middle dimension is related to the Euler characteristic by

$$e(X) = \sum_{i=0}^{n} (-1)^i \,\mathrm{h}^i(X) = \begin{cases} n + \mathrm{h}^n(X) & \text{if } n \text{ is even,} \\ n+1-\mathrm{h}^n(X) & \text{if } n \text{ is odd.} \end{cases}$$

From the cohomological results obtained above, we see that the only interesting intermediate Jacobian of an odd dimensional, smooth, projective hypersurface X is the one in odd, middle degree n, which we will denote shortly $\mathrm{J}\, X$.

Lemma 1.14 *The Hermitian form*

$$\widetilde{\mathcal{H}_X} \;:\; (\alpha,\beta) \mapsto \sum_{p+q=n} (-1)^{\frac{n^2-1}{2}+q} \cdot 2i \int_X \Pi^{p,q}\alpha \wedge \overline{\Pi^{p,q}\beta}$$

on $\mathrm{H}^n(X,\underline{\mathbb{C}})$ *is positive-definite when restricted to either half of the Hodge diamond of* X.

Proof The Hermitian form $\mathcal{H}_X$ in (1.5) decomposes as $\sum_{p+q=n} \mathcal{H}_X(\Pi^{p,q}(\cdot),\Pi^{p,q}(\cdot))$, where $\Pi^{p,q}$ is the projection to $\mathrm{H}^{p,q}(X)$. By Lemma 1.13, middle cohomology $\mathrm{H}^n(X,\underline{\mathbb{C}})$ is entirely primitive, so Lemma 1.5 gives us the sign corrections needed to make $\mathcal{H}_X$ positive on each summand. $\square$

Corollary 1.15 *If the middle Hodge numbers of an odd dimensional, smooth, projective hypersurface X vanish in periods of two, meaning:*

$$\forall 0 \leq l < n/2 \text{ even}: \mathrm{h}^{l,n-l}(X) = 0 \quad \text{or} \quad \forall 1 \leq l < n/2 \text{ odd}: \mathrm{h}^{l,n-l}(X) = 0, \tag{1.11}$$

then, the Hermitian form $\widetilde{\mathcal{H}_X}$ from Lemma 1.14 has as imaginary part $\Im\widetilde{\mathcal{H}_X} = \pm\Im\mathcal{H}_X$ an integer-valued, unimodular alternating form on $\rho\,\mathrm{H}^n(X,\mathbb{Z})$, making the pair $(\mathrm{J}^n X, \widetilde{\mathcal{H}_X})$ a principally polarized Abelian variety.

Lemma 1.16 *If $d > 1$ and $\mathbb{P}^l \subseteq X$ is a linear subspace, then $l \leq n/2$.*

Proof Let $Y \subseteq X$ be an l-dimensional subvariety. If $l > n/2$, the last map in

$$\mathrm{H}_{2l}(Y,\mathbb{Z}) \longrightarrow \mathrm{H}_{2l}(X,\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^{2n-2l}(X,\mathbb{Z}) \xleftarrow{\sim} \mathrm{H}^{2n-2l}(\mathbb{P}^{n+1},\mathbb{Z})$$

is an isomorphism. Then, Lemma 1.13 says that the fundamental class $[Y] \in \mathrm{H}^{2n-2l}(X,\mathbb{Z})$ equals a multiple $m \cdot [H]^{n-l}|_X$ of the pullback of a hyperplane class $[H]^{n-l} \in \mathrm{H}^{2n-2l}(\mathbb{P}^{n+1},\mathbb{Z})$. Then it holds

$$[H]^l|_Y(\{Y\}) = [Y] \smile [H]^l|_X(\{X\}) = m[H]^{n-l}|_X \smile [H]^l|_X(\{X\}) = md \overset{d>1}{\neq} 1.$$

On the other hand, any linear $\mathbb{P}^l \subseteq \mathbb{P}^{n+1}$ satisfies $[H]^l|_{\mathbb{P}^l}(\{\mathbb{P}^l\}) = 1$, so Y cannot be such.□

Chapter 2

The Cubic Threefold

Let V be a complex vector space of dimension 5. In this chapter we consider a smooth cubic threefold $X = Z(F) \subset \mathbb{P}V$, given by a cubic form $F \in \mathrm{H}^0(\mathbb{P}V, \mathcal{O}(3))$. Throughout we will denote $\mathcal{S}_{\mathbb{P}V}$ the tautological line bundle on $\mathbb{P}V$ and $\mathcal{S}_X$ its restriction to X.

2.1 Cohomology

By the Lefschetz hyperplane theorem employed in Lemma 1.13, it only remains to compute the middle row of the Hodge diamond of X. From the Euler sequence for the tangent bundle $0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}V}|_X \to \mathcal{O}_{\mathbb{P}V}(X)|_X \to 0$ we compute the total Chern class

$$\begin{aligned}\mathrm{c}(\mathcal{T}_X) = \sum_{i=0}^{3} c_i(\mathcal{T}_X) = \mathrm{c}(\mathcal{T}_{\mathbb{P}V}|_X) \cdot \mathrm{c}(\mathcal{O}_X(3))^{-1} = \mathrm{c}(\mathcal{O}_X(1))^5 \cdot \mathrm{c}(\mathcal{O}_X(3))^{-1} \\ = \frac{(1+[H])^5}{(1+3[H])} = (1 - 3[H] + 9[H]^2 - 27[H]^3) \cdot \sum_{i=0}^{3} \binom{5}{i} [H]^i,\end{aligned}$$

and so by the Hirzebruch-Riemann-Roch theorem we obtain

$$\begin{aligned}e(X) = \int_X \mathrm{c}_3(\mathcal{T}_X) = \int_X \left(\binom{5}{3} - 3 \cdot \binom{5}{2} + 9 \cdot 5 - 27 \right) [H]^3 \\ = (10 - 30 + 45 - 27) \int_X [H]^3 = -2 \cdot 3 = -6.\end{aligned}$$

With $\mathrm{h}^{2,1} = \mathrm{h}^{1,2}(X) = \frac{1}{2}\mathrm{h}^3(X) = \frac{1}{2}(4 - e(X)) = 5$, we have completed the Hodge diamond:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 5 & & 5 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The intermediate Jacobian $(\mathrm{H}^{1,2}(X)/\rho\ \mathrm{H}^3(X,\mathbb{Z}), \mathcal{H}_X)$ is thus a principally polarized Abelian five-fold.

2.2 Geometry of lines

The threefold does not contain any plane $\mathbb{P}^2 \subset X$ by Lemma 1.16, but the geometry of its lines is very rich, as we are about to see. For a more general discussion about Fano varieties of linear subspaces in projective hypersurfaces we recommend [Huy23, **2**.1].

Lemma 2.1 *Through every point of X there passes at least one line and at most 6. Through a generic point there pass exactly 6 lines.*

Proof Let $a \in X$, and $\mathbb{T}_a X = Z(\partial_\bullet F(a)) \subset \mathbb{P}V$ the embedded tangent hyperplane. The cubic surface $X \cap \mathbb{T}_a X \subset \mathbb{T}_a X$ intersects any line $L \subset \mathbb{T}_a X$ in a with multiplicity ≥ 2. If L is not contained in X, by Bézout's theorem, it intersects X either in a with multiplicity 3 or in a second point $b \in \mathbb{T}_a X$ with multiplicity 1. So for every $b \in X \cap \mathbb{T}_a X \setminus \{a\}$:

$$\begin{aligned} \text{the line } \langle a,b\rangle \text{ is contained in } X \iff \langle a,b\rangle \text{ is tangent to } X \text{ at } b \iff \langle a,b\rangle \subset \mathbb{T}_b X \\ \iff a \in \mathbb{T}_b X = Z(\partial_\bullet F(b)) \iff \partial_a F(b) = 0 \iff b \in Z(\partial_a F). \end{aligned}$$

Therefore, the intersection
$$X \cap \mathbb{T}_a X \cap Z(\partial_a F) \subset \mathbb{T}_a X$$
is the union of lines in X passing through a. Take a plane $H_a \subset \mathbb{T}_a X$ not passing through a, so that it parametrizes lines in $\mathbb{T}_a X$ passing through a. Since $\partial_a F$ is a quadratic polynomial (defined up to scaling), again by Bézout's theorem, the intersection $X \cap Z(\partial_a F) \cap H_a \subset H_a \cong \mathbb{P}^2$ consists of 6 points b, counting multiplicity, which trace a line $\langle a,b\rangle$ contained in X.

Fix now a hyperplane $H \subset \mathbb{P}V$. Then $H_a = H \cap \mathbb{T}_a X$ is an admissible choice of plane in $\mathbb{T}_a X$ for all $a \in X \setminus H$. The set of such a whose 6 associated points $b \in H_a$ are not distinct is cut out by the resultant of the two polynomials F and $\partial_a F$ with respect to the coefficients of a. Thus, for generic $a \in X$, the 6 lines through a are distinct. $\square$

Proposition 2.2 *For every line $L \subset X$, from the projectivization of the restricted tangent bundle $\mathcal{T}_X|_L$ there exists a dominant rational map*

$$\mathbb{P}(\mathcal{T}_X|_L) \dashrightarrow X$$

generically finite of degree 2. Since $\mathbb{P}(\mathcal{T}_X|_L)$ is birational to $\mathbb{P}^3$, this implies that X is unirational.

Proof We follow [Huy23, **II**.1.21]. For any $(a, [v]) \in \mathbb{P}(\mathcal{T}_X|_L)$, the direction v determines a line $L_{(a,v)} := \langle v \rangle \subset \mathbb{T}_a X \subset \mathbb{P}V$ which intersects X in a with tangent direction v. We construct the map as follows. If $L_{(a,[v])}$ is not one of the 6 lines contained in X, by Bézout's theorem, it intersects X either in a with multiplicity 3, or in a unique other point b with multiplicity 1. Assigning to $(a, [v])$ this third point $b(a, [v])$, potentially equal to a, defines a regular map from an open dense subset $U \subset \mathbb{P}(\mathcal{T}_X|_L)$.

The fiber over a $y \in X \setminus L$ is given by all the lines L' in the 2-dimensional linear span $\langle L, y \rangle$, such that L' is tangent at X at the point of intersection $L \cap L'$. This happens if and only if, set theoretically,

$$\{y\} \cup (L \cap L') = L' \cap X = L' \cap (X \cap \langle L, y \rangle).$$

Since X cannot contain a plane by Lemma 1.16, the intersection $X \cap \langle L, y \rangle$ is a cubic curve, and L is a degree 1 component of it. Then, the residual curve $C_y \subset \langle L, y \rangle$ is a conic, either irreducible or a union of two distinct lines (possibly including L). Thus, a line L' such as before must intersect y and a point of $C_y \cap L \subset \langle L, y \rangle$. If $L \not\subset C_y$, there are either one or two such distinct points.

Observe now that a generic plane $\mathbb{P}^2 \subset \mathbb{P}V$ containing L intersects X along L with multiplicity 1, meaning that the residual conic in $\mathbb{P}^2 \cap X$ does not contain L. Concluding, for a generic $y \in X \setminus L$ there are either one or two points $(a, [v]) \in U$ with $b(a, [v]) = y$. In particular, the rational map thus defined dominates X. $\square$

Lemma 2.3 *X is rationally connected.*

Proof As in the proof of Proposition 2.2, for any line $L \subset X$ and $y \in X \setminus L$, the reducible cubic curve $X \cap \langle L, y \rangle$ connects y to a point on L by either a line or a conic. $\square$

Remark 2.4 Lemma 2.3 says that $\mathrm{CH}_0(X) = \mathbb{Z}$. Applying [BS83, 1.(ii)], one can prove that the Abel-Jacobi map is an isomorphism: $\mathrm{CH}_1(X)_{\mathrm{alg}} \xrightarrow{\sim} \mathrm{CH}_1(X)_{\mathrm{hom}} \xrightarrow{\sim} \mathrm{J}^3 X$, but the result is outside of the scope of this thesis.

Remark 2.5 For any $p \in X$, projection from p to a hyperplane $\mathbb{P}^3 \subset \mathbb{P}V$ induces a degree 2 rational map

$$X \overset{\deg 2}{\dashrightarrow\!\!\!\!\rightarrow} \mathbb{P}^3 .$$

Thus, the degree of irrationality is $[X : \mathbb{C}(x_1, x_2, x_3)] \leq 2$.

Let $L = \mathbb{P}W \subset X$ be a line spanned by a 2-dimensional subspace $W \subset V$. Denote $\pi : V \twoheadrightarrow V/W$ the quotient map. With the choice of a splitting $V = W \oplus W'$ we can think of $\mathbb{P}(V/W)$ as the embedded $\mathbb{P}W' \subset \mathbb{P}V$ not intersecting $\mathbb{P}W$, but it is more natural to keep working invariantly. Consider the rational projection $\mathbb{P}V \dashrightarrow \mathbb{P}(V/W)$ induced by π. The fiber of $X \dashrightarrow \mathbb{P}(V/W)$ over a point a is $(X \setminus L) \cap \langle L, a \rangle$. Since X cannot contain the whole 2-dimensional projective plane $\langle L, a \rangle$, the intersection $X \cap \langle L, a \rangle$ is a degree 3 planar curve, and L is a degree 1 component of it. Thus, the residual curve of L in $X \cap \langle L, a \rangle$ is a possibly non-reduced conic. The rational projection can be extended to a morphism by blowing up at L:

$$\begin{tikzcd}
\mathrm{Bl}_L X \arrow[r, hook] \arrow[d, two heads] & \mathrm{Bl}_L \mathbb{P}V \arrow[d, two heads, "\tau"] \arrow[dr, two heads] & \\
X \arrow[r, hook] & \mathbb{P}V \arrow[r, dashed] & \mathbb{P}(V/W) .
\end{tikzcd} \tag{2.1}$$

Proposition 2.6 *The projection $\varphi : \mathrm{Bl}_L X \twoheadrightarrow \mathbb{P}(V/W)$ is a bundle of plane conics. The set of points $a \in \mathbb{P}(V/W)$ such that the fiber $\varphi^{-1}(a)$ is not smooth is a curve of degree* 5*, called the discriminant curve $D_L \subset \mathbb{P}(V/W)$. For every $a \in D_L$, the reducible conic $\varphi^{-1}(a)$ is*

- *a union of two distinct lines $L_1 \cup L_2 \subset \mathrm{Bl}_L X$ coplanar with L, if a is a smooth point of D_L,*
- *a scheme-theoretic double line, if a is a singular point of D_L.*

Before delving into the proof, we describe $\mathrm{Bl}_L X$ geometrically in terms of vector bundles. On $\mathbb{P}(V/W)$ there is the vector bundle $\mathcal{J}$ parametrizing 3-dimensional subspaces of V containing W, such that over each $a \in \mathbb{P}(V/W)$ the fiber is $\mathcal{J}|_a = \pi^{-1}(a)$; it fits with the tautological line bundle $\mathcal{S}_{\mathbb{P}(V/W)}$ in the exact sequences

$$\begin{tikzcd}
\mathbb{P}(V/W) \times W \arrow[r, hook] \arrow[d, equal] & \mathcal{J} \arrow[r, two heads] \arrow[d, hook] & \mathcal{S}_{\mathbb{P}(V/W)} \arrow[d, hook] \\
\mathbb{P}(V/W) \times W \arrow[r, hook] & \mathbb{P}(V/W) \times V \arrow[r, two heads] & \mathbb{P}(V/W) \times V/W
\end{tikzcd}$$

The blow-up $\mathrm{Bl}_{\mathbb{P}W} \mathbb{P}V$ can be characterized as the projectivized tautological bundle $\mathbb{P}\mathcal{J}$, which naturally embeds as a subvariety into $\mathbb{P}V \times \mathbb{P}(V/W)$. The blow-up morphism τ and the projection $\psi : \mathrm{Bl}_{\mathbb{P}W} \mathbb{P}V \twoheadrightarrow \mathbb{P}(V/W)$ are simply the projections onto the factors $\mathbb{P}V$ and $\mathbb{P}(V/W)$, and the exceptional divisor of $\mathrm{Bl}_{\mathbb{P}W} \mathbb{P}V$ corresponds to $E := \mathbb{P}W \times \mathbb{P}(V/W)$. It is then a matter of unraveling definitions that the pullback $\tau^*\mathcal{S}_{\mathbb{P}V}$ of the tautological line bundle on $\mathbb{P}V$ is the relative tautological bundle of $\psi^*\mathcal{J}$ on $\mathbb{P}\mathcal{J}$:

$$\begin{tikzcd}
& \tau^*\mathcal{S}_{\mathbb{P}V} \arrow[r, hook] \arrow[dr, two heads] & \psi^*\mathcal{J} \arrow[d] \arrow[dr, two heads] & \\
E \arrow[r, hook] \arrow[dd, two heads] & \mathrm{Bl}_{\mathbb{P}W} \mathbb{P}V \arrow[r, equal] \arrow[dd, "\tau"] & \mathbb{P}\mathcal{J} \arrow[d, hook] \arrow[ddl, bend right] \arrow[ddr, bend left, "\psi"] & \mathcal{J} \arrow[dd, two heads] \\
& & \mathbb{P}V \times \mathbb{P}(V/W) \arrow[dl, two heads] \arrow[dr, two heads] & \\
\mathbb{P}W \arrow[r, hook] & \mathbb{P}V \arrow[rr, dashed] & & \mathbb{P}(V/W)
\end{tikzcd} \tag{2.2}$$

Proof We follow [Huy23, **II**.5.1] and [Bea77, 1.2]. Since $\tau^*\mathcal{S}_{\mathbb{P}V} \hookrightarrow \psi^*\mathcal{J}$ is the relative tautological bundle, the induced map on global sections of its dual, and any symmetric power thereof, is an isomorphism:

$$\mathrm{H}^0(\mathbb{P}\mathcal{J}, \mathrm{S}^3\, \tau^*\mathcal{S}^*_{\mathbb{P}V}) \xleftarrow{\sim} \mathrm{H}^0(\mathbb{P}\mathcal{J}, \mathrm{S}^3\, \psi^*\mathcal{J}^*) = \psi^*\, \mathrm{H}^0(\mathbb{P}(V/W), \mathrm{S}^3\, \mathcal{J}^*).$$

Recall the cubic form $F \in \mathrm{H}^0(\mathbb{P}V, \mathrm{S}^3\, \mathcal{S}^*_{\mathbb{P}V})$ defining X, which vanishes on $L = \mathbb{P}W$, and write $\tau^*F = \psi^*G$ for a $G \in \mathrm{H}^0(\mathbb{P}(V/W), \mathrm{S}^3\, \mathcal{J}^*)$. Since the restriction of G to the subbundle $\mathbb{P}(V/W) \times \mathrm{S}^3\, W \subset \mathrm{S}^3\, \mathcal{J}$ vanishes, by the exact sequence

$$\mathcal{S}^*_{\mathbb{P}(V/W)} \otimes S^2\mathcal{J}^* \hookrightarrow \mathrm{S}^3\, \mathcal{J}^* \twoheadrightarrow \mathbb{P}(V/W) \times \mathrm{S}^3\, W^* \tag{2.3}$$

we know that G actually comes from a section in $\mathrm{H}^0(\mathbb{P}(V/W), \mathcal{S}^*_{\mathbb{P}(V/W)} \otimes S^2\mathcal{J}^*)$.

We emphasize not to confuse ψ with its restriction φ to $\mathrm{Bl}_L X$. For $a \in \mathbb{P}(V/W)$, since $\varphi^{-1}(a)$ is the residual conic of L in $Z(F) \cap \mathbb{P}(\pi^{-1}(a))$ we have

$$\begin{tikzcd}
L \cup \varphi^{-1}(a) \arrow[r, equal] & Z(F) \cap \mathbb{P}(\pi^{-1}(a)) \arrow[r, phantom, "\subset"] & \mathbb{P}(\pi^{-1}(a)) \arrow[r, phantom, "\subset"] & \mathbb{P}V \\
& Z(\psi^* G|_a) \arrow[u, "\tau", "\wr"'] \arrow[r, phantom, "\subset"] & \mathbb{P}\mathcal{J}|_a \arrow[u, "\tau", "\wr"'] & \\
Z(H_a) \cup Z(Q_a) \arrow[r, equal] & Z(G|_a) \arrow[u, dashed] \arrow[r, phantom, "\subset"] & \mathcal{J}|_a \arrow[u, dashed] \arrow[r, phantom, "\subset"] & V, \arrow[uu, dashed]
\end{tikzcd}$$

where $0 \neq H_a \in \mathcal{J}|_a^*$ is a linear form vanishing on W and $0 \neq Q_a \in S^2\mathcal{J}|_a^*$ is a quadratic form such that $G|_a = H_a \otimes Q_a$. Then, necessarily, the conic

$$\begin{aligned}
\varphi^{-1}(a) = Z(\psi^* Q_a) \text{ is not smooth } \iff & \text{ the quadratic form } Q_a \text{ has rank} \leq 2 \\
& \iff \det Q_a = 0 \iff \det G|_a = 0\,,
\end{aligned}$$

since $\det H_a \neq 0$. The set D_L is thus cut out in $\mathbb{P}(V/W)$ by the zero set of the determinant section $\det G \in \mathrm{H}^0(\mathbb{P}(V/W), \det(\mathcal{S}^*_{\mathbb{P}(V/W)} \otimes S^2\mathcal{J}^*))$. From the exact sequence (2.3) we compute

$$\begin{aligned}
\det(\mathcal{S}^*_{\mathbb{P}(V/W)} \otimes S^2\mathcal{J}^*) &\cong \det \mathcal{S}^*_{\mathbb{P}(V/W)} \otimes (\det \mathcal{J}^*)^{\otimes 1+\operatorname{rank}\mathcal{J}} \\
&\cong \mathcal{S}^*_{\mathbb{P}(V/W)} \otimes (\det \mathcal{S}^*_{\mathbb{P}(V/W)} \otimes \det(\mathbb{P}(V/W) \times W^*))^{\otimes 4} \cong (\mathcal{S}^*_{\mathbb{P}(V/W)})^{\otimes 5}.
\end{aligned}$$

Thus, the discriminant locus $D_L = Z(\det G) \subset \mathbb{P}(V/W)$ is a curve of degree 5.

To study the singularities of D_L we have to work locally. Choose a trivializing open neighbourhood $U \subset \mathbb{P}(V/W)$ for $\mathcal{J}$, coordinates x_1, x_2 for U and y_1, y_2, y_3 for the fibers of $\mathcal{J}|_U$, meaning that $\mathrm{S}^\bullet \mathcal{J}|_U^* \cong \mathcal{O}_U[y_1, y_2, y_3]$. Write

$$G|_U = (h \cdot s) \cdot Q, \quad Q = \sum_{ij} q_{ij} \cdot y_i y_j, \quad h, q_{ij} \in \mathcal{O}_{\mathbb{P}(V/W)}(U),$$

where $s \in \mathcal{J}^*(U)$ denotes a non-vanishing section coming from $\mathcal{S}^*_{\mathbb{P}(V/W)}(U)$, with h likewise non-vanishing. Then we have the description

$$\mathrm{Bl}_L X \cap \psi^{-1}(U) = Z(\psi^* Q) \subset \psi^{-1}(U) \subset \mathrm{Bl}_L \mathbb{P}V.$$

Take now $a \in D_L \cap U$. Since the quadratic form has rank ≤ 2 at a, after a change of coordinates we can assume that $q_{ij}(a) = 0$ for $i \neq j$ and $q_{33}(a) = 0$. For $k = 1, 2$ we compute the partial derivative

$$\begin{aligned}(\partial_{x_k} \det G|_U)(a) &= h(a) \cdot (\partial_{x_k} \det(q_{ij})_{ij})(a) \\ &= h(a) \cdot \sum_\sigma \operatorname{sgn}\sigma \sum_j (\partial_{x_k} q_{j\,\sigma j})(a) \prod_{i \neq j} q_{i\,\sigma i}(a) = h(a) q_{11}(a) q_{22}(a)(\partial_{x_k} q_{33})(a).\end{aligned}$$

We claim that at least one of $(\partial_{x_1} q_{33})(a)$ or $(\partial_{x_2} q_{33})(a)$ is not 0. To see this, take the singular point $z = [0 : 0 : 1] \in Z(\psi^* Q|_a) \subset \mathbb{P}\mathcal{J}|_a$ of the degenerate conic, which satisfies $(\partial_{\overline{y_k}} \psi^* Q)(z) = 0$ for $\overline{y_1}, \overline{y_2}, \overline{y_3}$ the projective coordinates induced on $\mathbb{P}\mathcal{J}|_U$. Note that the pullbacks $\psi^* x_1$, $\psi^* x_2$ together with $\overline{y_1}, \overline{y_2}, \overline{y_3}$ form a generating set of coordinates for $\psi^{-1}(U)$. Then, with the crucial assumption that X and thus $\mathrm{Bl}_L X$ are smooth, at least one of the remaining partial derivatives cannot vanish:

$$0 \neq (\partial_{\psi^* x_k} \psi^* Q)(z) = \left(\sum_{ij} \partial_{x_k} q_{ij}(a) y_i y_j \right)(z) = (\partial_{x_k} q_{33})(a) \quad \text{for } k = 1 \text{ or } 2.$$

Therefore, we conclude that the reducible conic

$$\begin{aligned}&\varphi^{-1}(a) = Z(\psi^* Q|_a) \text{ is a union of two distinct lines} \\ &\iff \text{ the quadratic form } Q|_a \text{ has rank } 2 \iff q_{11}(a) \neq 0 \text{ and } q_{22}(a) \neq 0 \\ &\iff (\partial_{x_k} \det G|_U)(a) \neq 0 \text{ for } k = 1 \text{ or } 2 \iff a \in D_L \text{ is smooth.} \qquad \square\end{aligned}$$

Motivated by Proposition 2.6, we capture with the following definition the lines which induce a conic bundle with smooth discriminant curve. We will see in Proposition 2.15 that a generic line in the cubic threefold is of this type.

Definition 2.7 *A line $L \subset X$ is called of* first type *if for every plane $\mathbb{P}^2 \subset \mathbb{P}V$ containing L, the component of L is reduced in the scheme-theoretic intersection $\mathbb{P}^2 \cap X$.*

A line $L \subset X$ is called of second type *if there is a plane $\mathbb{P}^2 \subset \mathbb{P}V$ containing L which is tangent to X at each point of L. In this case, such a plane is unique.*

Lemma 2.8 *For every line $L \subset X$, the normal bundle of L in X is*

$$\mathcal{N}_{L/X} \cong \begin{cases} \mathcal{O}_L \oplus \mathcal{O}_L & \textit{if } L \textit{ is of first type} \\ \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1) & \textit{if } L \textit{ is of second type} \end{cases}$$

Proof The normal bundle sequences for $L \subset \mathbb{P}V$ and $X \subset \mathbb{P}V$ fit in the following commutative diagram, where every composition along a straight line is exact. [1]

$$
\begin{tikzcd}
 & & \mathcal{T}_L \arrow[dl, hook'] \arrow[dd, hook] & \\
 & \mathcal{T}_X|_L \arrow[dl, two heads] \arrow[dr, hook] & & \\
 & & \mathcal{T}_{\mathbb{P}V}|_L \arrow[d, two heads] \arrow[dr, two heads] & \\
\mathcal{N}_{L/X} \arrow[rr, hook] & & \mathcal{N}_{L/\mathbb{P}V} \arrow[r, two heads] & \mathcal{N}_{X/\mathbb{P}V}|_L
\end{tikzcd}
\tag{2.4}
$$

The normal bundle of the smooth, complete intersections L and X in $\mathbb{P}V$ is known to be isomorphic to $\mathcal{O}_L(1)^{\oplus 3}$ and $\mathcal{O}_L(3)$ respectively. Applying Grothendieck's theorem to the rank 2 bundle $\mathcal{N}_{L/X}$ on the curve L, we obtain integers a, b with $\mathcal{N}_{L/\mathbb{P}V} \cong \mathcal{O}_L(a) \oplus \mathcal{O}_L(b)$. The inclusion into $\mathcal{O}_L(1)^{\oplus 3}$ forces $a, b \leq 1$, whereas the exactness of the horizontal sequence forces

$$a + b = \deg(\mathcal{N}_{L/X}) = \deg(\mathcal{N}_{L/\mathbb{P}V}) - \deg(\mathcal{N}_{X/\mathbb{P}V}) = 3 - 3 = 0\,.$$

Thus, the only two cases are $(a, b) = (0, 0)$ or $(1, -1)$.

To distinguish the two cases according to Definition 2.7, consider a plane $\mathbb{P}^2 \subset \mathbb{P}V$ containing L. Then, $\mathbb{P}^2$ is tangent to X at each point of L if and only if the composition $\mathcal{N}_{L/\mathbb{P}^2} \to \mathcal{N}_{X/\mathbb{P}V}|_L$ is 0, which happens if and only if it factors through an inclusion in $\mathcal{N}_{L/X}$.

$$
\begin{tikzcd}
\mathcal{N}_{L/X} \arrow[r, hook] & \mathcal{N}_{L/\mathbb{P}V} \arrow[r, two heads] & \mathcal{N}_{X/\mathbb{P}V}|_L \\
 & \mathcal{N}_{L/\mathbb{P}^2} \arrow[ul, hook, dashed] \arrow[u, hook] \arrow[ur] &
\end{tikzcd}
$$

Knowing that $\mathcal{N}_{L/\mathbb{P}^2} \cong \mathcal{O}_L(1)$, if L is of second type, the factorization implies that $\mathcal{N}_{L/X}$ contains an $\mathcal{O}_L(1)$ summand. Conversely, if $\mathcal{N}_{L/X}$ contains an $\mathcal{O}_L(1)$ summand, then the 2-dimensional space of global sections $\mathrm{H}^0(L, \mathcal{N}_{L/X}) \cong \mathrm{H}^0(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(1))$ defines a plane $\mathbb{P}^2 \subset \mathbb{P}V$ containing L and tangent to X at each point of L, so L is of second type. $\square$

[1] The smallest Pythagorean triple which gives integer coordinates for all the objects in diagram (2.4), such that the short exact sequences stay straight lines, is $(45, 60, 75)$; we used the triple $(3, 4, 5)$ as an approximation.

2.3 Fano surface of lines

In this section, we construct the variety that parametrizes linear subspaces of dimension 1 lying in the cubic threefold X, called the Fano variety of lines. It will be convenient to start with the functorial approach and investigate concrete geometrical properties later. So let us consider X and $\mathbb{P}V$ scheme-theoretically.

Denote $\underline{\mathcal{S}ch}_{\mathbb{C}}$ the category of finite type complex schemes. By a *family of lines of X parametrized by $T \in \underline{\mathcal{S}ch}_{\mathbb{C}}$* we will mean a closed subscheme $Y \hookrightarrow X \times_{\mathbb{C}} T$ which is flat over T and such that, for each $t \in T$, the fiber

$$Y|_t = Y \times_{\mathbb{C}} \operatorname{Spec}(k(t)) \hookrightarrow X \times_{\mathbb{C}} \operatorname{Spec}(k(t)) \hookrightarrow \mathbb{P}V \times_{\mathbb{C}} \operatorname{Spec}(k(t))$$

is a linear subvariety of dimension 1.

Theorem 2.9 *The Fano functor of lines $\mathfrak{Fan}_X : \underline{\mathcal{S}ch}_{\mathbb{C}} \to \mathcal{S}ets$, which associates to each scheme T the set*

$$\{Y \subset X \times_{\mathbb{C}} T \mid Y \text{ is a family of lines of } X \text{ parametrized by } T\},$$

is representable by a connected, projective scheme over $\mathbb{C}$, which we will denote $\mathrm{F}X$.

Proof Every family of lines of X parametrized by a scheme T is the projectivization of a (unique up to isomorphism) rank 2 locally free subsheaf $\mathcal{F} \hookrightarrow \mathcal{O}_T^{\oplus 5}$ with locally free cokernel. Thus $\mathfrak{Fan}_X$ is naturally a subfunctor of the Grassmann functor, associating to each scheme T the set of isomorphism classes of all such subsheaves. The latter is representable by a smooth, integral scheme of dimension $2(5-2) = 6$, called the Grassmann variety $\mathbb{G} := \mathrm{Gr}_2(V)$, which naturally embeds into $\mathbb{P}(\bigwedge^2 V)$ via the Plücker embedding. We will show that a closed subscheme of $\mathbb{G}$ represents the subfunctor $\mathfrak{Fan}_X$. Consider the tautological rank 2 subsheaf $\mathcal{S}_{\mathbb{G}} \hookrightarrow \mathcal{O}_{\mathbb{G}} \otimes V$, and the induced restriction on symmetric powers

$$\mathcal{O}_{\mathbb{G}} \otimes \mathrm{S}^3 V^* \twoheadrightarrow \mathcal{O}_{\mathbb{G}}(3),$$

The cubic form $F \in \mathrm{S}^3 V^*$ defining X gives a global section $s_F \in \mathrm{H}^0(\mathbb{G}, \mathcal{O}_{\mathbb{G}}(3))$. Then, the subsheaf $\tau^*\mathcal{S}_{\mathbb{G}} \hookrightarrow \mathcal{O}_T \otimes V$ corresponding to a morphism $\tau : T \to \mathbb{G}$ induces a family of lines of X if and only if s_F maps to 0 under the restriction map τ^*.

$$\begin{tikzcd}
\mathrm{H}^0(T, \tau^*\mathcal{O}_{\mathbb{G}}(3)) \cong \mathrm{H}^0(T, \tau^*\mathcal{O}_{\mathbb{G}}(3)) & & \mathrm{H}^0(\mathbb{G}, \mathcal{O}_{\mathbb{G}}(3)) \arrow[ll, "\tau^*"'] \arrow[dl, "0 \mapsfrom s_F"] \\
& \mathrm{H}^0(Z(s_F), \mathcal{O}_{\mathbb{G}}(3)) \arrow[ul, dashed] &
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[rr, "\tau"] \arrow[dr, dashed] & & \mathbb{G} \\
& Z(s_F) \arrow[ur] &
\end{tikzcd}$$

In other words, $\mathfrak{Fan}_X$ corresponds to the subfunctor of $\mathbb{G}$ associating to each scheme T the set of morphisms $\tau : T \to \mathbb{G}$ such that $\tau^* s_F = 0$, which is representable by the closed subscheme $Z(s_F) \subset \mathbb{G}$ cut out by the section s_F. Moreover, since s_F is a section of a locally free sheaf with $\operatorname{rank} \mathrm{S}^3 \mathcal{S}_{\mathbb{G}}^* + 1 < \dim \mathbb{G}$, the closed subscheme $Z(s_F)$ is connected by the Fulton-Hansen connectedness theorem. □

Remark 2.10 In the proof above we have appealed to the existence of the Grassmann variety and then identified within it the closed subscheme representing the Fano functor. This process can be generalized to construct the Hilbert and Quot schemes, which represent the functor of arbitrary closed subschemes of a projective scheme and the functor of arbitrary quotients of a flat, coherent sheaf. The Fano functor is naturally a subfunctor of the Hilbert functor $\mathfrak{Hilb}_X$ of closed subschemes of X, which is representable by the projective scheme Hilb_X over $\mathbb{C}$. A family of closed subschemes $Y \in X \times_{\mathbb{C}} T \subset \mathbb{P}V \times_{\mathbb{C}} T$ is a family of lines if and only if it has constant Hilbert polynomial $z+1$ on every fiber, thus $\mathfrak{Fan}_X$ is representable by the closed stratum $\mathrm{Hilb}_X^{z+1} \subset \mathrm{Hilb}_X$.

Theorem 2.11 *The Fano scheme* $\mathrm{F}X$ *is a smooth variety of dimension* 2.

Proof Reducedness and irreducibility will follow from smoothness, since we already have connectedness. Let us briefly recall the deformation-theoretic description of the tangent space $\mathrm{T}_L \, \mathrm{F}X$ at a line $L \subset X$. Tangent vectors correspond to first order deformations, which are morphisms of schemes

$$\operatorname{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2) \to \mathrm{F}X, \quad (\varepsilon) \mapsto L.$$

By the defining property of the Fano scheme, such a morphism corresponds to a family of lines

$$L_\varepsilon \subset X \times_{\mathbb{C}} \operatorname{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2) =: X_\varepsilon, \quad L = L_\varepsilon|_{\varepsilon=0} \subset X_\varepsilon|_{\varepsilon=0} = X\,,$$

so to exact sequences of sheaves on X_ε

$$\mathcal{I}_{L_\varepsilon} \hookrightarrow \mathcal{O}_{X_\varepsilon} \twoheadrightarrow \mathcal{O}_{L_\varepsilon}$$

such that $\mathcal{O}_{L_\varepsilon}$ is a flat $\mathbb{C}[\varepsilon]/\varepsilon^2$-module and the pullback through $\mathbb{C}[\varepsilon]/\varepsilon^2 \twoheadrightarrow \mathbb{C}$ recovers the original sequence on $L \subset X$. An $\mathcal{O}_{X_\varepsilon}$-module $\mathcal{F}$ is flat over $\mathbb{C}[\varepsilon]/\varepsilon^2$ if and only if its tensor product with the unique non-trivial ideal $(\varepsilon) \hookrightarrow \mathbb{C}[\varepsilon]/\varepsilon^2$ preserves left-exactness, that is if $(\varepsilon) \otimes \mathcal{F} \to \mathcal{F}$ is injective. It then holds by the five lemma that the kernel $\mathcal{K} = \ker(\mathcal{O}_{X_\varepsilon} \twoheadrightarrow$

$\mathcal{F}$) is also flat. So the first order deformations of L in X correspond to flat quotient sheaves $\mathcal{O}_{X_\varepsilon} \twoheadrightarrow \mathcal{F}$ fitting in the commutative diagram

$$\begin{tikzcd}
\mathbb{C}\otimes(\varepsilon) \arrow[dr, "\sim"] & & \mathcal{I}_L\otimes(\varepsilon) \arrow[rr, hook] \arrow[dr, "\sim"] & & \mathcal{O}_X\otimes(\varepsilon) \arrow[rr, two heads] \arrow[dr, "\sim"] & & \mathcal{O}_L\otimes(\varepsilon) \arrow[dr, "\sim"] & \\
& (\varepsilon) \arrow[d, hook] & & \mathcal{K}\otimes(\varepsilon) \arrow[rr, hook] \arrow[d, hook] & & \mathcal{O}_{X_\varepsilon}\otimes(\varepsilon) \arrow[rr, two heads] \arrow[d, hook] & & \mathcal{F}\otimes(\varepsilon) \arrow[d, hook] \\
& \mathbb{C}[\varepsilon]/\varepsilon^2 \arrow[d, two heads] & & \mathcal{K} \arrow[rr, hook] \arrow[d, two heads] & & \mathcal{O}_{X_\varepsilon} \arrow[rr, two heads] \arrow[d, two heads] & & \mathcal{F} \arrow[d, two heads] \\
& \mathbb{C} & & \mathcal{I}_L \arrow[rr, hook] & & \mathcal{O}_X \arrow[rr, two heads] & & \mathcal{O}_L\,.
\end{tikzcd}$$

From any such diagram one can take the connecting homomorphism

$$\mathcal{I}_L \to \mathcal{O}_{X_\varepsilon}/\mathcal{I}_L\otimes(\varepsilon) \to \mathcal{O}_L\otimes(\varepsilon) \xrightarrow{\sim} \mathcal{O}_L\,,$$

and it is a standard deformation theory exercise that any sheaf homomorphism $\mathcal{I}_L \to \mathcal{O}_L$ induces such a diagram. Thus, first order deformations of L in X correspond to global sections of the sheaf

$$\mathcal{H}om(\mathcal{I}_L,\mathcal{O}_L) \cong \mathcal{H}om(\mathcal{I}_L/\mathcal{I}_L^2,\mathcal{O}_L) = \mathcal{N}_{L/X},$$

and thus $\mathrm{T}_L\,\mathrm{F}X \cong \mathrm{H}^0(L,\mathcal{N}_{L/X})$ has dimension 2 in both cases of Lemma 2.8.

A similar argument shows that first order deformations of L in X are unobstructed if

$$\mathrm{H}^1(L,\mathcal{N}_{L/X}) = 0\,,$$

we refer to [Huy23, **II**.1.3] for a detailed discussion. This is satisfied for both lines of first and second type, as we have seen in Lemma 2.8, since $\mathrm{H}^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(\pm 1)) = 0$. Therefore, the Fano scheme is smooth at every point and of dimension 2. $\square$

Since $\mathrm{F}X$ is smooth, going further, we will consider it simply as a complex manifold. The restriction to the Fano surface of the tautological bundle sequence on the Grassmannian gives the vector bundles

$$\mathcal{S}_{\mathrm{F}X} \hookrightarrow \mathcal{O}_{\mathrm{F}X}\otimes V \twoheadrightarrow \mathcal{Q}_{\mathrm{F}X}$$

with $\mathcal{S}_{\mathrm{F}X}|_{\mathbb{P}W} \cong W$ and $\mathcal{Q}_{\mathrm{F}X}|_{\mathbb{P}W} \cong V/W$ for every $\mathbb{P}W \in \mathrm{F}X$. The projectivization $\mathbb{L} := \mathbb{P}(\mathcal{S}_{\mathrm{F}X}) \hookrightarrow (\mathrm{F}X)\times\mathbb{P}V$ is the tautological $\mathbb{P}^1$-bundle over $\mathrm{F}X$ with $\mathbb{L}|_L \cong L$ for every $L \in \mathrm{F}X$, and its image in $(\mathrm{F}X)\times\mathbb{P}V$ is the incidence correspondence of lines. The two projections on $(\mathrm{F}X)\times X$ restrict to projections p and q on $\mathbb{L}$ giving diagram (2.5), which we will refer to as the *Fano correspondence*.

Lemma 2.12 *The projection $q : \mathbb{L} \to X$ is a flat, finite morphism of degree* 6.

Proof It is finite of generic degree 6 by Lemma 2.1. Moreover, this holds for every fiber because any two points are rationally equivalent, by Lemma 2.3. This implies that q is also flat. $\square$

$$\begin{tikzcd}
(\mathrm{FX}) \times X \supset \{(L,x) \mid x \in L\} & \mathbb{L} \arrow[l, hook', "\sim"'] \arrow[d, two heads, "p", "\mathbb{P}^1\text{-bundle}"'] := \mathbb{P}\mathcal{S}_{\mathrm{FX}} \arrow[r, two heads, "q", "\deg 6"'] & X \subset \mathbb{P}V \\
& \mathrm{FX} &
\end{tikzcd} \tag{2.5}$$

It is a matter of unraveling definitions that the relative tautological line bundle on $\mathbb{L}$ is the pullback of the tautological bundle on X:

$$q^* \mathcal{S}_X = \mathcal{S}_{\mathbb{L}} \hookrightarrow p^* \mathcal{S}_{\mathrm{FX}} \hookrightarrow \mathcal{O}_{\mathbb{L}} \otimes V \,. \tag{2.6}$$

We will see in Lemma 2.21 that q is actually the projective morphism induced by the line bundle $\mathcal{S}_{\mathbb{L}}^*$.

Now we want to distinguish the lines of first and second type within the Fano surface. The Gauss map $\mathcal{G}_X$ of the cubic threefold X is the morphism

$$\begin{tikzcd}
\mathbb{P}V \arrow[r, dashed, "p \mapsto (v \mapsto \partial_v F(p))"] & \mathbb{P}V^* \\
X \arrow[u, hook] \arrow[r, two heads] & \mathcal{G}_X(X) \arrow[u, hook] =: X^* \,,
\end{tikzcd}$$

well-defined everywhere because X is smooth. Being defined by the derivatives of a cubic polynomial, it is a finite morphism of degree 2. Geometrically, it sends a point $p \in X$ to the embedded tangent hyperplane $\mathbb{T}_p X = Z(\partial_\bullet F(p)) \subset \mathbb{P}V$ through the isomorphism

$$\ker : \mathbb{P}V^* \xrightarrow{\sim} \mathrm{Gr}(4, V) \,.$$

Lemma 2.13 *A line $L \subset X$ is of first type if and only if the restriction of the Gauss map*

$$\mathcal{G}_X : L \twoheadrightarrow \mathcal{G}_X(L)$$

is an isomorphism onto a smooth plane conic. It is of second type if and only if $\mathcal{G}_X : L \twoheadrightarrow \mathcal{G}_X(L)$ is a double cover onto a line.

Proof First we note that the image $\mathcal{G}_X(L)$ is contained in the dual variety $L^* \cong \mathbb{P}^2 \subset \mathbb{P}V^*$, and that planes $P \cong \mathbb{P}^2 \subset \mathbb{P}V$ containing L correspond to lines $P^* \cong \mathbb{P}^1 \subset \mathbb{P}V^*$ contained in L^*. Moreover, any plane $P \supset L$ is tangent to X at $p \in L$ if and only if $\mathcal{G}_X(p) \in P^*$. Thus, L is of first type if and only if $\mathcal{G}_X(L)$ is not contained in any line. Choosing coordinates such that the Gauss map simplifies to a regular map $\mathbb{P}^1 \to \mathbb{P}^2$ defined by polynomials of degree 2, it becomes clear that it is an isomorphism if and only if the image is a smooth plane conic, and is a double cover onto a line otherwise. □

We will need the following important corollary of the reflexivity theorem, which applies to any smooth hypersurface in projective space over an algebraically closed field of characteristic 0; for the general statement and proof, see [EH16, 10.21].

Theorem 2.14 (Reflexivity) *The Gauss map $\mathcal{G}_X : X \to X^*$ is a finite and birational morphism. In particular, it is generically injective.*

Proposition 2.15 *The subset of lines of second type in the Fano surface, which we denote $\mathcal{D}_2 \subset \mathrm{F}X$, is a curve. In particular, a generic line $L \in \mathrm{F}X$ is of first type.*

Proof Using Theorem 2.14, if $S \subset X$ is a proper, Zariski-closed subset such that $\mathcal{G}_X$ is injective on $X \setminus S$, then any line $L \subset X$ not contained in S must be of the first type. Indeed, except for the finite subset of points $L \cap S$, the restriction $\mathcal{G}_X|_L$ must be injective, so it can only be an isomorphism by Lemma 2.13. So a generic line is of the first type.

To distinguish the type of a line $L \subset X$ with an algebraic criterion on $\mathrm{F}X$ we look at the twisted normal bundle:

$$\dim \mathrm{H}^1(L, \mathcal{N}_{L/X}(-1)) = \begin{cases} 0 & \text{if } L \text{ is of first type} \\ 1 & \text{if } L \text{ is of second type.} \end{cases}$$

Consider the cubic form $F \in \mathrm{S}^3 V^*$ and three linear forms $l_1, l_2, l_3 \in V^*$ such that $L = Z(l_1, l_2, l_3) = \mathbb{P}W$. The Euler sequence sends W and V isomorphically onto the global sections of $\mathcal{T}_L(-1)$ and $\mathcal{T}_{\mathbb{P}}|_L(-1)$, and it also shows that the higher cohomology of $\mathcal{T}_L(-1)$ vanishes. Together with the isomorphisms $\mathcal{O}_L(1)^{\oplus 3} \xrightarrow{\sim} \mathcal{N}_{L/\mathbb{P}}$ and $\mathcal{O}_L(3) \xrightarrow{\sim} \mathcal{N}_{X/\mathbb{P}}|_L$, we obtain the following diagram of cohomology groups.

$$
\begin{tikzcd}
W \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{H}^0(L, \mathcal{T}_L(-1)) \arrow[d, hook] & & \\
V \arrow[r, "\sim"] \arrow[d, two heads] & \mathrm{H}^0(L, \mathcal{T}_{\mathbb{P}}|_L(-1)) \arrow[d, two heads] \arrow[dr] & & \\
V/W \arrow[r, "\sim"] \arrow[dr, "\overline{v} \mapsto (\partial_v l_i)_i"'] \arrow[drr, "\overline{v} \mapsto \partial_v F" near end] & \mathrm{H}^0(L, \mathcal{N}_{L/\mathbb{P}}(-1)) \arrow[r] & \mathrm{H}^0(L, \mathcal{N}_{X/\mathbb{P}}|_L(-1)) \arrow[r] & \mathrm{H}^1(L, \mathcal{N}_{L/X}(-1)) \\
 & \mathrm{H}^0(L, \mathcal{O}_L^{\oplus 3}) \arrow[r] \arrow[u, "\wr"] & \mathrm{H}^0(L, \mathcal{O}_L(2)) \arrow[u, "\wr"] & \mathrm{S}^2 W^* \arrow[l, hook', "\sim"']
\end{tikzcd}
\tag{2.7}
$$

By exactness of the horizontal sequence at $\mathcal{N}_{X/\mathbb{P}}|_L(-1)$, we see that L is of first type if and only if the morphism $\overline{v} \in V/W \mapsto \partial_v F \in \mathrm{S}^2 W^*$ has full rank 3; it is well defined because $\partial_w F$ vanishes identically for $w \in W$.

Now we globalize this criterion using the tautological bundles on the Fano surface. Note that the morphism above is just the restriction over a fiber of the holomorphic map

$$
\psi : (L, \overline{v}) \in \mathcal{Q}_{\mathrm{FX}} \mapsto (L, \partial_v F) \in \mathrm{S}^2 \mathcal{S}_{\mathrm{FX}}^*,
$$

which is not a vector bundle homomorphism. Then, the subset of lines of second type is precisely the degeneracy locus

$$
\mathcal{D}_2 = \{L \in \mathrm{FX} \mid \operatorname{rank} \psi|_L < 3\} = Z(\det \psi) \subset \mathrm{FX},
$$

where $\det \psi$ is the determinant section in $\mathrm{H}^0(L, \det(\mathcal{Q}_{\mathrm{FX}}^* \otimes \mathrm{S}^2 \mathcal{S}_{\mathrm{FX}}^*))$. $\square$

Remark 2.16 Let us draw a connection between the surjectivity of $\psi|_{\mathbb{P}W} : V/W \to \mathrm{S}^2 W^*$ discussed in the proof and the non-degeneracy of the Gauss map, which allows one to bypass the argument involving the normal bundle sequences. Surjectivity is equivalent to the injectivity of the dual

$$
\mathrm{S}^2 W \xrightarrow{\sim} (\mathrm{S}^2 W^*)^* \to (V/W)^* \hookrightarrow V^*, \quad p \mapsto (v \mapsto \partial_v F(p)).
$$

After precomposing with the Veronese embedding, the projectivization of this dual homomorphism is precisely the Gauss map $\mathcal{G}_X$ restricted to the line $\mathbb{P}W \subset \mathbb{P}V$.

$$
\begin{tikzcd}
V \arrow[r, "v\mapsto\partial_v F"] \arrow[d, two heads] & \mathrm{S}^2 V^* \arrow[d, two heads] \\
V/W \arrow[r, "\overline{v}\mapsto\partial_v F"] & \mathrm{S}^2 W^*
\end{tikzcd}
\qquad
\begin{tikzcd}
V^* & \mathrm{S}^2 V \arrow[l, "(v\mapsto\partial_v F(p))\mapsfrom p"'] \\
(V/W)^* \arrow[u, hook] & \mathrm{S}^2 W \arrow[l] \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{P}V^* & \mathbb{P}\,\mathrm{S}^2 V \arrow[l, dashed] & \mathbb{P}V \arrow[l, hook', "\text{Veronese}"] \arrow[ll, bend right, "\mathcal{G}_X"'] \\
\mathbb{P}(V/W)^* \arrow[u, hook] & \mathbb{P}\,\mathrm{S}^2 W \arrow[l, dashed] \arrow[u, hook] & \mathbb{P}W \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}
$$

Since the image of $\mathbb{P}W \hookrightarrow \mathbb{P}\,\mathrm{S}^2 W$ is not contained in a hyperplane, we obtain the following equivalences:

$$
\begin{aligned}
&\operatorname{im}(\mathrm{S}^2 W \to V^*) \text{ has dimension 3} \iff \operatorname{im}(\mathbb{P}\,\mathrm{S}^2 W \dashrightarrow \mathbb{P}V^*) \text{ is not contained in a line} \\
\iff\ &\mathcal{G}_X(\mathbb{P}W) = \operatorname{im}(\mathbb{P}W \to \mathbb{P}V^*) \text{ is not contained in a line} \iff \mathbb{P}W \text{ is of first type.}
\end{aligned}
$$

Remark 2.17 There is a morphism $\mathcal{D}_2 \to \mathrm{F}X$ mapping a line of second type $L \subset X$ to its residual L' (potentially equal to L) in the scheme-theoretic intersection $X \cap P = 2L \cup L'$, where P is the unique plane tangent to X along L. Its image $\mathcal{R}_2 \subset \mathrm{F}X$ is a curve, called the *residual curve* in the Fano surface. So a generic line $L \in \mathrm{F}X$ does not intersect any line of second type.

2.4 Conic bundles

By Proposition 2.6, every line $L = \mathbb{P}W \subset X$ of first type which does not intersect any line of second type realizes $\mathrm{Bl}_L X$ as a conic bundle over $\mathbb{P}(V/W)$ with a smooth discriminant curve D_L, such that for all $a \in D_L$ the reducible conic $\varphi^{-1}(a)$ is reduced. We fix such a line L for the rest of this section, which is generic because $\mathcal{D}_2 \cup \mathcal{R}_2 \subset \mathrm{F}X$ is a proper, Zariski-closed subset. Our discussion can be generalized to second type lines, but it will not be necessary to do so for our purposes in the subsequent chapters.

Consider diagram (2.8) below, which is the juxtaposition of the Fano correspondence and the blow-up diagram (4.1). The conic bundle φ has a section $D_L \hookrightarrow \varphi^{-1}(D_L)$ sending each point a to the unique singular point of the reducible conic $\varphi^{-1}(a)$. Blowing up $\varphi^{-1}(D_L)$ along this section has the effect of resolving the node at each fiber $\varphi^{-1}(a)$, giving a normalization $\widetilde{\varphi^{-1}(D_L)} \twoheadrightarrow D_L$ with each fiber made up of two distinct lines.

Since $\widetilde{\varphi^{-1}(D_L)}$ parametrizes a family of lines of X, it embeds into the incidence correspondence $\mathbb{L}$ by sending each point $\widetilde{x}$ over $\varphi(\widetilde{x}) \in D_L$ to the pair $(\tau(L'), \tau(\widetilde{x}))$ with $L' \subset \varphi^{-1}(\varphi(\widetilde{x}))$ the (unique after normalization) line containing $\widetilde{x}$. Call $C_L := p(\widetilde{\varphi^{-1}(D_L)})$

its image. Then, the projection φ factors through p to a morphism of curves $\overline{\varphi} : C_L \twoheadrightarrow D_L$ with fibers of constant length 2, thus an étale double cover.

$$
\begin{tikzcd}
 & \widetilde{\varphi^{-1}(D_L)} \arrow[r, two heads, "\text{normalization}"] \arrow[dl, hook'] \arrow[dd, "p"] & \varphi^{-1}(D_L) \arrow[r, hook] \arrow[dd, "\varphi"] & \mathrm{Bl}_L X \arrow[d, two heads, "\tau"] \arrow[dr, two heads, "\varphi"] & \\
\mathbb{L} \arrow[rrr, two heads, "q"] \arrow[d, two heads, "p"] & & & X \arrow[r, dashed] & \mathbb{P}(V/W) \\
\mathrm{FX} & C_L \arrow[l, hook', "\iota_L"'] \arrow[r, two heads, "\substack{\text{étale}\\ \text{double cover}}", "\overline{\varphi}"'] & D_L \arrow[urr, hook] & &
\end{tikzcd}
\tag{2.8}
$$

Lemma 2.18

$$C_L := p(\widetilde{\varphi^{-1}(D_L)}) = \{L' \in \mathrm{FX} \mid L' \neq L,\, L' \cap L \neq \emptyset\} \subset \mathrm{FX}$$

is a smooth, irreducible curve of genus 11. *The* $\mathbb{P}^1$*-bundle* $p : \widetilde{\varphi^{-1}(D_L)} \twoheadrightarrow C_L$ *has a section*

$$C_L \xrightarrow{\sim} \widetilde{C_L} := \{(L', x) \in \mathbb{L} \mid L' \cap L = \{x\}\} \subset \mathbb{L}, \quad L' \mapsto (L', L' \cap L).$$

The projection q *restricts to a degree* 5 *morphism* $\widetilde{C_L} \twoheadrightarrow L$.

Proof The subset description is immediate from the definitions, taking into account that L is of first type and thus none of the fibers of φ contain L as a component. Since p restricts to a smooth morphism from $\widetilde{\varphi^{-1}(D_L)}$ to its image C_L, the latter is also smooth.

The function $L' \in C_L \to L' \cap L \in X$ is a morphism, so its lift to the incidence correspondence is too, and the projection p is obviously an inverse morphism to it. Note that $q^{-1}(L) = p^{-1}(L) \sqcup \widetilde{C_L}$, so the fibers of the restriction $q|_{\widetilde{C_L}}$ have generic length 5.

If C_L is disconnected, then it must be 2 disjoint copies of D_L, since we have the étale double cover $\overline{\varphi}$. But then, the degree 5 restriction $q : C_L \twoheadrightarrow L$ would have to decompose in two morphisms $D_L \twoheadrightarrow L$ of degrees summing up to 5. Since D_L is not rational, and as a smooth plane quintic it is also not hyperelliptic, this is not possible. Alternatively, one could use the connectedness theorem [Har77, **III**.7.9], which applies to the divisor $C_L \subset FX$ because we will see in Lemma 2.29 that it is ample. The genus is determined by the Riemann-Hurwitz formula for the étale double cover onto the smooth curve D_L of genus 6. $\square$

Remark 2.19 Since lines in X are parametrized by the connected Fano surface, which is a smooth algebraic variety, they all represent algebraically and homologically equivalent cycles. So also the class $p_* q^* [L'] \in \mathrm{H}^2(\mathrm{FX}, \mathbb{Z})$ is indipendent of any particular line $L' \in \mathrm{FX}$. The next result gives a representative of this class in terms of $L \notin \mathcal{D}_2 \cup \mathcal{R}_2$.

Lemma 2.20 $p_* q^*[L] = [C_L] \in \mathrm{H}^2(\mathrm{F}X, \mathbb{Z})$

Proof Since q is flat, we can use that the pullback on cohomology coincides with the intersection-theoretic pullback:

$$q^*[L] = [q^{-1}(L)] = [p^{-1}(L) \sqcup \widetilde{C_L}] \in \mathrm{H}^2(\mathbb{L}, \mathbb{Z}).$$

By the definition of the pushforward p_* in terms of the Poincaré duality maps, it holds

$$p_*[p^{-1}(L) \sqcup \widetilde{C_L}] = p_*[p^{-1}(L)] + [p(\widetilde{C_L})] = [C_L],$$

where the first term vanishes because $p(p^{-1}(L)) = \{L\}$ is a point. □

Lemma 2.21 *The pullback $q^* : V^* \cong \mathrm{H}^0(\mathbb{P}V, \mathcal{S}^*_{\mathbb{P}V}) \hookrightarrow \mathrm{H}^0(\mathbb{L}, \mathcal{S}^*_{\mathbb{L}})$ is an isomorphism. Equivalently, $q : \mathbb{L} \to \mathbb{P}V$ is the morphism induced by the line bundle $\mathcal{S}^*_{\mathbb{L}}$:*

$$\begin{tikzcd} & \mathbb{P}(\mathrm{H}^0(\mathbb{L}, \mathcal{S}^*_{\mathbb{L}})^*) \arrow[d, "(q^*)^*", "\wr"'] \\ \mathbb{L} \arrow[ur, bend left, "\varphi_{\mathcal{S}^*_{\mathbb{L}}}"] \arrow[r, two heads, "q"] & X \subset \mathbb{P}V \cong \mathbb{P}(\mathrm{H}^0(\mathbb{P}V, \mathcal{S}^*_{\mathbb{P}V})^*) \end{tikzcd}$$

Proof The cohomology exact sequence for the twisted ideal sheaf of X in $\mathbb{P}V$ is

$$0 = \mathrm{H}^0(\mathbb{P}V, \mathrm{S}^2\,\mathcal{S}_{\mathbb{P}V}) \to \mathrm{H}^0(\mathbb{P}V, \mathcal{S}^*_{\mathbb{P}V}) \to \mathrm{H}^0(X, \mathcal{S}^*_X) \to \mathrm{H}^1(\mathbb{P}V, \mathrm{S}^2\,\mathcal{S}_{\mathbb{P}V}) = 0,$$

so the middle map is an isomorphism. Consider now the induced morphism $\varphi_{\mathcal{S}^*_{\mathbb{L}}}$, commuting with q through the (a priori rational) projection, and $\widetilde{X}$ its image in $\mathbb{P}(\mathrm{H}^0(\mathbb{L}, \mathcal{S}^*_{\mathbb{L}})^*)$. Then, $(q^*)^*$ restricts to a finite morphism $\widetilde{X} \twoheadrightarrow X$, which we will prove to be generically injective and thus an isomorphism, because X is normal. Assuming this, the claim follows from

$$q^* \;:\; V^* \cong \mathrm{H}^0(\mathbb{P}V, \mathcal{S}^*_{\mathbb{P}V}) \xrightarrow{\sim} \mathrm{H}^0(X, \mathcal{S}^*_X) \xrightarrow{\sim} \mathrm{H}^0(\widetilde{X}, \mathcal{S}^*_{\widetilde{X}}) \xleftarrow{\sim} \mathrm{H}^0(\mathbb{L}, \mathcal{S}^*_{\mathbb{L}}).$$

By Lemma 2.18, for a generic $L \in \mathrm{F}X$ the diagram restricts to

$$\begin{tikzcd} & \widetilde{L} \arrow[r, hook] \arrow[d, two heads, "(q^*)^*"] & \mathbb{P}(\mathrm{H}^0(\mathbb{L}, \mathcal{S}^*_{\mathbb{L}})^*) \arrow[d, "(q^*)^*", "\wr"'] \\ p^{-1}(L) \sqcup \widetilde{C_L} = q^{-1}(L) \arrow[ur, two heads, "\varphi_{\mathcal{S}^*_{\mathbb{L}}}"] \arrow[r, two heads, "q", "(\deg 1, \deg 5)"'] & L \subset \mathbb{P}V & \cong \mathbb{P}(\mathrm{H}^0(\mathbb{P}V, \mathcal{S}^*_{\mathbb{P}V})^*). \end{tikzcd} \tag{2.9}$$

Since 5 is prime and $\varphi_{\mathcal{S}^*_{\mathbb{L}}}$ cannot be of degree 1 onto a (disconnected) curve, as it is not an isomorphism, it must decompose in degree 1 and 5 morphisms. So $(q^*)^* : \widetilde{L} \to L$ an isomorphism, where $\widetilde{L}$ is the preimage of L under $(q^*)^*$. Thus $(q^*)^* : \widetilde{X} \to X$ is generically injective. □

2.5 Geometry of the Fano surface

In this section we study the Plücker embedding $\iota : \mathrm{FX} \hookrightarrow \mathbb{G} := \mathrm{Gr}_2(V) \hookrightarrow \mathbb{P}(\bigwedge^2 V)$ induced by the determinant bundle $\det \mathcal{S}^*_{\mathrm{FX}}$, which is the pullback of $\mathcal{S}^*_{\mathbb{P}(\wedge^2 V)}$ on $\mathbb{P}(\bigwedge^2 V)$. Since $\mathcal{S}_{\mathbb{L}} \hookrightarrow p^*\mathcal{S}_{\mathrm{FX}} \hookrightarrow \mathcal{O}_{\mathbb{L}} \otimes V$ is the relative tautological bundle (refer to the Fano correspondence diagram), it holds $\mathrm{H}^0(\mathbb{L}, \mathcal{S}^*_{\mathbb{L}}) \xleftarrow{\sim} \mathrm{H}^0(\mathbb{L}, p^*\mathcal{S}^*_{\mathrm{FX}})$. Together with Lemma 2.21 and the fact that $\mathrm{H}^0(\mathrm{FX}, \mathcal{S}^*_{\mathrm{FX}}) \overset{p^*}{\hookrightarrow} \mathrm{H}^0(\mathbb{L}, p^*\mathcal{S}^*_{\mathrm{FX}})$ because p^* is a fiber bundle, we obtain

$$
\begin{tikzcd}
\mathrm{H}^0(X,\mathcal{S}^*_X) \arrow[d, "q^*", "\wr"'] & & \mathrm{H}^0(\mathcal{O}_X \otimes V^*) \arrow[ll, "\sim"'] \arrow[d, "q^*", "\wr"'] & \\
\mathrm{H}^0(\mathbb{L},\mathcal{S}^*_{\mathbb{L}}) & \mathrm{H}^0(\mathbb{L},p^*\mathcal{S}^*_{\mathrm{FX}}) \arrow[l, "\sim"'] & \mathrm{H}^0(\mathbb{L},\mathcal{O}_{\mathbb{L}} \otimes V^*) \arrow[l, "\sim"'] & V^* \arrow[l, "\sim"'] \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\
 & \mathrm{H}^0(\mathrm{FX},\mathcal{S}^*_{\mathrm{FX}}) \arrow[u, "p^*", "\wr"'] & \mathrm{H}^0(\mathrm{FX},\mathcal{O}_{\mathrm{FX}} \otimes V^*) \arrow[l, "\sim"'] \arrow[u, "p^*", "\wr"'] & \\
 & \mathrm{H}^0(\mathbb{G},\mathcal{S}^*_{\mathbb{G}}) \arrow[u, "\wr"'] & \mathrm{H}^0(\mathbb{G},\mathcal{O}_{\mathbb{G}} \otimes V^*) \arrow[l, "\sim"'] \arrow[u, "\wr"'] &
\end{tikzcd}
\tag{2.10}
$$

It is known that $\mathbb{G} \hookrightarrow \mathbb{P}(\bigwedge^2 V)$ is non-degenerate [2] and linearly normal. [3] We will prove that also the Fano surface is non-degenerate and linearly normal, or equivalently, that the remaining maps in the following diagram are isomorphisms:

$$
\begin{tikzcd}
\bigwedge^2 V^* \arrow[r, "\sim"] \arrow[d, "\wr"'] & \bigwedge^2 \mathrm{H}^0(\mathbb{G},\mathcal{S}^*_{\mathbb{G}}) \arrow[r, "\sim"] \arrow[d, "\wr"', "\wedge"] & \bigwedge^2 \mathrm{H}^0(\mathrm{FX},\mathcal{S}^*_{\mathrm{FX}}) \arrow[d, "\wedge"] \\
\mathrm{H}^0(\mathbb{P}(\bigwedge^2 V),\mathcal{S}^*_{\mathbb{P}(\wedge^2 V)}) \arrow[r, "\sim"] & \mathrm{H}^0(\mathbb{G},\det \mathcal{S}^*_{\mathbb{G}}) \arrow[r] & \mathrm{H}^0(\mathrm{FX},\det \mathcal{S}^*_{\mathrm{FX}})
\end{tikzcd}
\tag{2.11}
$$

A hyperplane section of $\mathrm{FX} \subset \mathbb{P}(\bigwedge^2 V)$ is given by the vanishing of a section in $\mathrm{H}^0(\mathrm{FX}, \det \mathcal{S}^*_{\mathrm{FX}})$ coming from an alternating form $0 \neq \alpha \in \bigwedge^2 V^*$. The lines in a hyperplane $Z(\alpha)$ containing a point $\overline{v} \in \mathbb{P}V$ are

$$\{\mathbb{P}W \in \mathrm{FX} \cap Z(\alpha) \mid \overline{v} \in \mathbb{P}W\} = \left\{\mathbb{P}W \in \mathrm{FX} \,\middle|\, {\overline{v} \in \mathbb{P}W \atop \alpha(v,\bullet)|_W \equiv 0}\right\} = \left\{\mathbb{P}W \in \mathrm{FX} \,\middle|\, {\overline{v} \in \mathbb{P}W \atop \mathbb{P}W \subset P_{\alpha,\overline{v}}}\right\},$$

where $P_{\alpha,\overline{v}} := \mathbb{P}\ker(\alpha(v,\bullet))$ is the so-called polar subspace. Since $\alpha \neq 0$, the radical $K_\alpha := \ker(v \in V \mapsto \alpha(v,\bullet) \in V^*) \subset V$ is of codimension ≥ 1, so a generic point $\overline{v} \in \mathbb{P}V \setminus \mathbb{P}K_\alpha$ defines a 3-dimensional polar hyperplane $P_{\alpha,\overline{v}}$ which contains $\mathbb{P}K_\alpha \neq \emptyset$. Note

[2] A subvariety $Y \subset \mathbb{P}^n$ is called non-degenerate if it is not contained in any hyperplane.

[3] An embedding $Y \hookrightarrow \mathbb{P}^n$ is called linearly normal if for every rational projection $\mathbb{P}^N \dashrightarrow \mathbb{P}^n$ from a higher dimensional projective space, and every lift $Y \hookrightarrow \mathbb{P}^N$ which commutes with it, the image of Y in $\mathbb{P}^N$ is already contained in a hyperplane $\mathbb{P}^n \subset \mathbb{P}^N$.

that, since α is alternating, $K_\alpha \subset V$ is of even codimension 2 or 4, according to wether it is decomposable as a $\lambda_1 \wedge \lambda_2$, and in this case $K_\alpha = \ker \lambda_1 \cap \ker \lambda_2$ and the hyperplane section is

$$\mathrm{FX} \cap Z(\alpha) = \left\{ L \in \mathrm{FX} \,\middle|\, {\scriptstyle \exists x \in L \atop L \subset P_{\alpha,x}} \right\} = \{ L \in \mathrm{FX} \mid L \cap \mathbb{P}K_\alpha \neq \emptyset \},$$

because the plane $\mathbb{P}K_\alpha \subset P_{\alpha,x}$ must intersect any line $L \subset P_{\alpha,x}$. Conversely, every plane $P \subset \mathbb{P}V$ gives rise to such a hyperplane section.

Lemma 2.22 *For all $L \in \mathrm{FX} \setminus (\mathcal{D}_2 \cup \mathcal{R}_2)$ and $0 \neq \alpha \in \bigwedge^2 V^*$ it holds*

$$C_L \subset Z(\alpha) \iff L \subset \mathbb{P}K_\alpha .$$

Proof If $L \subseteq \mathbb{P}K_\alpha$, then every $L' \in C_L$ crosses $\mathbb{P}K_\alpha$ and is thus in $Z(\alpha)$. For the converse, we consider two cases. If $\mathbb{P}K_\alpha \cap L \neq \emptyset$, then, by choosing homogeneous coordinates $L = \langle x_1, x_2 \rangle$ with $x_1 \in \mathbb{P}K_\alpha$, one checks that the polar hyperplanes $P_{\alpha,\varepsilon_1 x_1 + \varepsilon_2 x_2} = P_{\alpha, x_2} \cong \mathbb{P}^3$ are constant along $\varepsilon_1 x_1 + \varepsilon_2 x_2 \in L \setminus \mathbb{P}K_\alpha$. If $\mathbb{P}K_\alpha \cap L = \emptyset$, then the polars along L all contain the $\geq$ 2-dimensional subspace $\langle L, \mathbb{P}K_\alpha \rangle \subseteq \mathbb{P}V$, and thus lie on a line in the dual space $\mathbb{P}V^*$. Since L is of first type, by Lemma 2.13 the image of the Gauss map $\mathcal{G}(L) \subset \mathbb{P}V^*$ does not lie on a line, so a generic embedded tangent space $\mathbb{T}_x X$, for $x \in L$, is different from $P_{\alpha,x}$. Moreover, a generic $\mathbb{T}_x X$ is linearly spanned by the lines in X through x. So there exist lines in C_L not contained in the polar of their intersection point with L, and thus not contained in $Z(\alpha)$. □

Many of the following results were already known to Fano [Fan04]; we take the opportunitiy to present his proofs in a modern light.

Lemma 2.23 (Fano) *Let $L, L', L'' \in \mathrm{FX} \setminus (\mathcal{D}_2 \cup \mathcal{R}_2)$ be pairwise non-intersecting lines. Then, for the projective linear spans of the curves $C_L, C_{L'}, C_{L''}$ in $\mathbb{P}(\bigwedge^2 V)$ holds*

$$\dim\langle C_L \cup C_{L'} \rangle = 9,\ \dim\langle C_L \rangle = 6,\ \dim(\langle C_L \rangle \cap \langle C_{L'} \rangle) = 3,\ \dim(\langle C_L \rangle \cap \langle C_{L'} \rangle \cap \langle C_{L''} \rangle) = 0.$$

Proof The linear span is alternatively given by the hyperplane intersection

$$\langle C_L \rangle = \bigcap_{\substack{\alpha \in \wedge^2 V^* \\ C_L \subset Z(\alpha)}} Z(\alpha) = \bigcap_{\substack{\alpha \in \wedge^2 V^* \\ L \subseteq \mathbb{P}K_\alpha}} Z(\alpha) = \bigcap_{\substack{\lambda_1, \lambda_2 \in V^* \\ L \subseteq \mathbb{P}(\ker \lambda_1 \cap \ker \lambda_2)}} Z(\lambda_1 \wedge \lambda_2) = Z(U_L),$$

where $U_L = \left\langle \left\{ \lambda_1 \wedge \lambda_2 \,\middle|\, {\scriptstyle \lambda_1, \lambda_2 \in V^* \atop L \subseteq \mathbb{P}(\ker \lambda_1 \cap \ker \lambda_2)} \right\} \right\rangle \subset \bigwedge^2 V^*$ is a 3-dimensional subspace spanned by the decomposable tensors whose radical contains L. This proves $\dim\langle C_L \rangle = 9 - 3 = 6$. Moreover, it holds $\langle C_L \rangle \cap \langle C_{L'} \rangle = Z(U_L \oplus U_{L'})$, $\langle C_L \rangle \cap \langle C_{L'} \rangle \cap \langle C_{L''} \rangle = Z(U_L \oplus U_{L'} \oplus U_{L''})$, and one checks that $(U_L \oplus U_{L'}) \cap U_{L''} = 0$ because the lines are non-intersecting. So the remaining equalities in the claim follow by linear algebra:

$$\dim(U_L \cap U_{L'}) = 0, \quad \dim(U_L \oplus U_{L'}) = 6, \quad \dim(U_L \oplus U_{L'} \oplus U_{L''}) = 9. \qquad \square$$

Proposition 2.24 (Fano) $\mathrm{F}X \hookrightarrow \mathbb{P}(\bigwedge^2 V)$ *is non-degenerate, and its degree is* 45. *The curves* $C_L \subset \mathbb{P}(\bigwedge^2 V)$ *for* $L \in \mathrm{F}X \setminus (\mathcal{D}_2 \cup \mathcal{R}_2)$ *have degree* 15. *In particular,* $\mathrm{H}^0(\mathbb{P}(\bigwedge^2 V), \mathcal{S}^*_{\mathbb{P}(\wedge^2 V)}) \to \mathrm{H}^0(\mathrm{F}X, \det \mathcal{S}^*_{\mathrm{F}X})$ *is injective.*

Proof Non-degeneracy follows already from the first equality in Lemma 2.23, but one can give a more direct argument: if $\mathrm{F}X$ is contained in a hyperplane $Z(\alpha)$, then a generic tangent space $\mathbb{T}_x X$ (which is spanned by the lines in X through x) equals to the polar hyperplane $P_{\alpha,x}$ and thus contains $\mathbb{P}K_\alpha \neq \emptyset$, which is not the case because the image of the Gauss map $X^* \subset \mathbb{P}V^*$ is itself non-degenerate (here one uses the reflexivity theorem [EH16, 10.21]).

For the degrees, it is convenient to choose hyperplanes cut out by decomposable tensors. For two generic planes $P_1, P_2 \subset \mathbb{P}V$ whose intersection is a line $P_1 \cap P_2$, the cubic surface $X \cap \langle P_1, P_2 \rangle$ is smooth and $X \cap P_1 \cap P_2$ consists of 3 distinct points not lying on any line of $X \cap \langle P_1, P_2 \rangle$ and with 6 distinct lines in X through each. Taking decomposable tensors α_1, α_2 whose radical planes P_1, P_2 are as above, it holds

$$\mathrm{F}X \cap Z(\alpha_1, \alpha_2) = \left\{ L \in \mathrm{F}X \,\middle|\, \begin{smallmatrix} L \cap P_1 \neq \emptyset \\ L \cap P_2 \neq \emptyset \end{smallmatrix} \right\} = \left\{ L \in \mathrm{F}X \,\middle|\, \begin{smallmatrix} \text{either } L \subset X \cap \langle P_1, P_2 \rangle \\ \text{or } L \cap P_1 \cap P_2 \neq \emptyset \end{smallmatrix} \right\}.$$

There are 27 lines in $X \cap \langle P_1, P_2 \rangle$ and $3 \cdot 6$ lines intersecting $X \cap P_1 \cap P_2$, thus 45 in total.

For planes P_1, P_2 as above, and such that $X \cap P_1 = L \cup L' \cup L''$ are 3 distinct lines not in $\mathcal{D}_2 \cup \mathcal{R}_2$, we have

$$\mathrm{F}X \cap Z(\alpha_1, \alpha_2) = (C_L \cap Z(\alpha_2)) \cup (C_{L'} \cap Z(\alpha_2)) \cup (C_{L''} \cap Z(\alpha_2)).$$

By symmetry, it follows that each hyperplane section $C_L \cap Z(\alpha_2)$ has $\frac{45}{3} = 15$ lines. $\square$

Proposition 2.25 (Fano) $\mathrm{F}X \hookrightarrow \mathbb{P}(\bigwedge^2 V)$ *is linearly normal. , and thus* $\mathrm{H}^0(\mathbb{P}(\bigwedge^2 V), \mathcal{S}^*_{\mathbb{P}(\wedge^2 V)}) \to \mathrm{H}^0(\mathrm{F}X, \det \mathcal{S}^*_{\mathrm{F}X})$ *is surjective.*

Proof Let $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}(\bigwedge^2 V)$ be a rational projection from a higher dimensional projective space and $\widetilde{\mathrm{F}X} \subset \mathbb{P}^N$ a surface, such that π restricts to an isomorphism $\widetilde{\mathrm{F}X} \xrightarrow{\sim} \mathrm{F}X$. For every $L \in \mathrm{FX} \setminus (\mathcal{D}_2 \cup \mathcal{R}_2)$, denote $\widetilde{C_L} \subset \widetilde{\mathrm{FX}}$ the preimage of C_L. Then, since every line in X intersects at least one line not in $\mathcal{D}_2 \cup \mathcal{R}_2$, we have

$$\widetilde{\mathrm{FX}} = \bigcup_{L \in \mathrm{FX} \setminus (\mathcal{D}_2 \cup \mathcal{R}_2)} \widetilde{C_L} \quad \subseteq \langle \ (\widetilde{C_L})_{L \notin \mathcal{D}_2 \cup \mathcal{R}_2} \ \rangle \subset \mathbb{P}^N.$$

Since the projection is linear, it restricts to surjections $\langle \widetilde{C_L} \rangle \twoheadrightarrow \langle C_L \rangle$. However, the smooth curves $\widetilde{C_L} \subset \mathbb{P}^N$ of degree 15 and genus 11 satisfy $\dim \langle \widetilde{C_L} \rangle \leq 6$ by general curve theory; the maximal genus of a non-degenerate degree 15 curve in $\mathbb{P}^7$ is 10, by Castelnuovo's bound [ACGH85, III.2]. So for any other curves $C_{L'}, C_{L''}$ we obtain isomorphisms

$$\begin{array}{ccccc}
\langle \widetilde{C_L} \rangle \cap \langle \widetilde{C'_L} \rangle \cap \langle \widetilde{C''_L} \rangle & \subset & \langle \widetilde{C_L} \rangle \cap \langle \widetilde{C'_L} \rangle & \subset & \langle \widetilde{C_L} \rangle \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
\langle C_L \rangle \cap \langle C'_L \rangle \cap \langle C''_L \rangle & \subset & \langle C_L \rangle \cap \langle C'_L \rangle & \subset & \langle C_L \rangle
\end{array}$$

Thus, the curves $\widetilde{C_L}, \widetilde{C_{L'}}$ of any two non-intersecting L, L' span together a $\mathbb{P}^9$ inside of which lies every other curve $\widetilde{C_{L''}}$, and thus the whole $\widetilde{\mathrm{FX}}$. $\square$

With the following result we improve upon the know fact that the Grassmannian is the intersection of all the quadrics containing it:

$$\mathrm{Gr}_2(V) = \bigcap_{\substack{Q \subset \mathbb{P}(\bigwedge^2 V) \text{ quadric} \\ \mathrm{Gr}_2(V) \subseteq Q}} Q \quad \subset \mathbb{P}(\textstyle\bigwedge^2 V).$$

Lemma 2.26 (Fano) *Every quadric $Q \subset \mathbb{P}(\bigwedge^2 V)$ containing* FX *also contains* $\mathrm{Gr}_2(V)$.

Proof First we show that $Q \supset \mathrm{F}X$ contains every line $L \subset \mathbb{P}V$ triply tangent to X at a generic $x \in X$, meaning that $x \in L \cap X$ is of multiplicity 3. Denote $C_x \subset \mathrm{Gr}_2(V)$ the subvariety of lines passing through $x \in X$ and contained in $\mathbb{T}_x X$. Then, the elements of $C_x \cap Q$ span set-theoretically either the whole embedded tangent hyperplane $\mathbb{T}_x X$ or a quadric cone therein which contains every line in X through x by assumption. On the other hand, the lines triply tangent to X at x form a quadric cone $K_x \subset \mathbb{T}_x X$, which in particular contains every line in X through x. Since through a generic x there pass ≥ 5 distinct lines in X, the cone $K_x \subset \mathbb{T}_x X$ is uniquely determined by these lines and must therefore be contained in the cone spanned by the elements of $C_x \cap Q$. So $C_x \cap Q$ also contains all triple tangents at such a generic x.

Let now $P \subset \mathbb{P}V$ be a generic plane such that the cubic curve $X \cap P$ is smooth. A line $L \subset P$ is triply tangent to X at $x \in L \cap X$ if and only if $L = \mathbb{T}_x(X \cap P)$ is the embedded tangent line at an inflection point x. The Gauss map

$$\begin{tikzcd} P \arrow[r, dashed, "x \mapsto Z(\partial_\bullet F(x))"] & P^* & \subset & \mathrm{Gr}_2(V) \\ X \cap P \arrow[u, hook] \arrow[r, two heads, "x \mapsto \mathbb{T}_x(X \cap P)"] & (X \cap P)^* \arrow[u, hook] \end{tikzcd}$$

is ramified precisely at the inflection points, so the triply tangent lines are singular points of the dual curve $(X \cap P)^*$. However, a smooth cubic curve has precisely 9 inflection points, so if $Q \cap P^* \subset P^* \subset \mathrm{Gr}_2(V)$ were a conic containing all of them (and each with multiplicity ≥ 2 because they are singular points), then their intersection product

$$Q \cdot (X \cap P)^* \geq 2 \cdot 9 > 2 \cdot 6 = \deg Q \cdot \deg(X \cap P^*)$$

would contradict Bézout's theorem. So Q must contain all of P^*, and since P was generic also all of $\mathrm{Gr}_2(V)$. $\square$

The fact that $\mathrm{Gr}_2(V) \subset \mathbb{P}(\bigwedge^2 V)$ is the intersection of all quadrics containing $\mathrm{F}X$ makes it in some sense canonical and places severe restrictions on what vector bundles can induce an equivalent Plücker embedding.

Theorem 2.27 *Let $\mathcal{F}$ be a rank 2 vector bundle on $\mathrm{F}X$, such that $\mathcal{F}^*$ is globally generated,* [4] *$\mathrm{h}^0(\mathrm{F}X, \mathcal{F}^*) = 5$ and there is a vector bundle isomorphism $f : \det \mathcal{F}^* \xrightarrow{\sim} \det \mathcal{S}^*_{\mathrm{F}X}$. Then, there is a vector bundle isomorphism $g : \mathcal{F}^* \xrightarrow{\sim} \mathcal{S}^*_{\mathrm{F}X}$ such that $g \wedge g = f$.*

Proof We follow [Tyu70]. First we show that the alternating product

$$\textstyle\bigwedge^2 \mathrm{H}^0(\mathrm{F}X, \mathcal{F}^*) \longrightarrow \mathrm{H}^0(\mathrm{F}X, \det \mathcal{F}^*) \tag{2.12}$$

is an isomorphism. Consider the projectivization $\pi : \mathbb{P}\mathcal{F} \twoheadrightarrow F$ and its relative tautological bundle $\mathcal{S}_{\mathbb{P}\mathcal{F}} \hookrightarrow \pi^*\mathcal{F}$, so that $\mathrm{H}^0(\mathcal{F}^*) \xrightarrow{\sim} \mathrm{H}^0(\mathcal{S}^*_{\mathbb{P}\mathcal{F}})$ and the dual $\mathcal{S}^*_{\mathbb{P}\mathcal{F}}$ is globally generated. The induced morphism $\varphi_{\mathcal{S}^*_{\mathbb{P}\mathcal{F}}}$ sends the fibers $\pi^{-1}(L)$ over $L \in \mathrm{F}X$ to lines, so there is an induced morphism $\varphi_{\mathcal{F}^*}$ to the Grassmannian, see diagram (2.13). This morphism commutes with the Plücker embedding $\mathrm{Gr}_2(\mathrm{H}^0(\mathcal{F}^*)^*) \hookrightarrow \mathbb{P}(\bigwedge^2 \mathrm{H}^0(\mathcal{F}^*)^*)$, the rational map r induced by the dual of (2.12), and the induced morphism $\varphi_{\det \mathcal{F}^*}$. The fact that f is an isomorphism of bundles means that $\varphi_{\det \mathcal{F}^*}$ commutes with the pullback $f^* : \mathrm{H}^0(\det \mathcal{S}^*_{\mathrm{F}X}) \xrightarrow{\sim} \mathrm{H}^0(\det \mathcal{F}^*)$

[4] A vector bundle $\mathcal{F}$ is called *globally generated* if each fiber $\mathcal{F}|_p$ is generated by global sections.

and $\varphi_{\det \mathcal{S}^*_{\mathrm{FX}}}$, which is none other than the Plücker embdedding $\mathrm{F}X \subset \mathrm{Gr}_2(V) \subset \mathbb{P}(\bigwedge^2 V)$. Let $H \subset \mathbb{P}(\mathrm{H}^0(\det \mathcal{F}^*)^*)$ be a maximal linear subspace such that r restricts to an embedding $\widetilde{r} : H \hookrightarrow \mathbb{P}(\bigwedge^2 \mathrm{H}^0(\mathcal{F}^*)^*)$, and let C be a center for the projection r with respect to H. We want to show that C is empty, meaning that r is an isomorphism.

$$\begin{tikzcd}
\mathbb{P}\mathcal{F} \arrow[r, "\varphi_{\mathcal{S}^*_{\mathbb{P}\mathcal{F}}}"] \arrow[d, "\pi"'] & \mathbb{P}(\mathrm{H}^0(\mathcal{S}^*_{\mathbb{P}\mathcal{F}})^*) \arrow[r, "\sim"] & \mathbb{P}(\mathrm{H}^0(\mathcal{F}^*)^*) & \\
\mathrm{FX} \arrow[r, "\varphi_{\mathcal{F}^*} : L \mapsto \varphi_{\mathcal{S}^*_{\mathbb{P}\mathcal{F}}}(\pi^{-1}(L))"] \arrow[dd, hook] \arrow[ddr, bend right, "\varphi_{\det \mathcal{S}^*_{\mathrm{FX}}}"'] \arrow[ddrr, "\varphi_{\det \mathcal{F}^*}"] & \mathrm{Gr}_2(\mathrm{H}^0(\mathcal{S}^*_{\mathbb{P}\mathcal{F}})^*) \arrow[r, "\sim"] & \mathrm{Gr}_2(\mathrm{H}^0(\mathcal{F}^*)^*) \arrow[d, hook] & \\
 & & \mathbb{P}(\bigwedge^2 \mathrm{H}^0(\mathcal{F}^*)^*) & \\
\mathrm{Gr}_2(V) \arrow[r, hook] & \mathbb{P}(\mathrm{H}^0(\det \mathcal{S}^*_{\mathrm{FX}})^*) & \mathbb{P}(\mathrm{H}^0(\det \mathcal{F}^*)^*) \arrow[l, "\sim", "(f^*)^*"'] \arrow[u, dashed, "r"'] \arrow[r, phantom, "\subset"] & H \arrow[ul, "\widetilde{r}"']
\end{tikzcd} \tag{2.13}$$

Then, if we denote F' and F'' the images of $\mathrm{F}X$ in $\mathbb{P}(\mathrm{H}^0(\det \mathcal{F}^*)^*)$ and $\mathbb{P}(\bigwedge^2 \mathrm{H}^0(\mathcal{F}^*)^*)$ respectively, set-theoretically it holds

$$\forall Q \subseteq \mathbb{P}(\textstyle\bigwedge^2 \mathrm{H}^0(\mathcal{F}^*)^*) : \mathrm{F}'' \subseteq Q \implies \mathrm{F}' \subseteq K(C, \widetilde{r}^{-1}(Q)),$$

where $K(C, \widetilde{r}^{-1}(Q)) := \bigcup_{q \in \widetilde{r}^{-1}(Q)} \langle C, q \rangle$ is the cone over $\widetilde{r}^{-1}(Q)$ with center C. In particular, for every quadric $Q \subset \mathbb{P}(\bigwedge^2 \mathrm{H}^0(\mathcal{F}^*)^*)$ containing $\mathrm{Gr}_2(\mathrm{H}^0(\mathcal{F}^*)^*)$, the quadric cone $K(C, \widetilde{r}^{-1}(Q))$ contains F' and thus the whole $\mathrm{Gr}_2(V)$ by Lemma 2.26.

If $\dim C \geq 1$, we could have chosen C generic such that there do not exist any quadrics at all in $\mathbb{P}(\bigwedge^2 V)$ which contain $\mathrm{Gr}_2(V)$ and contain C in their cone, so this case does not occur and $\operatorname{codim} H \leq 1$. The function

$$\left\{ Q \subset \mathbb{P}(\textstyle\bigwedge^2 \mathrm{H}^0(\mathcal{F}^*)^*) \,\middle|\, \substack{Q \text{ quadric} \\ \mathrm{Gr}_2(\mathrm{H}^0(\mathcal{F}^*)^*) \subset Q} \right\} \xrightarrow{Q \mapsto K(C, \widetilde{r}^{-1}(Q))} \left\{ Q \subset \mathbb{P}(\textstyle\bigwedge^2 V) \,\middle|\, \substack{Q \text{ quadric} \\ \mathrm{Gr}_2(V) \subset Q} \right\} \tag{2.14}$$

is linear, by viewing quadrics as points in the symmetric power $\mathbb{P}(\mathrm{S}^2(\bigwedge^2 \mathrm{H}^0(\mathcal{F}^*)))$ of the dual projective spaces. It is also injective, because if $\widetilde{r}^{-1}(Q) = \widetilde{r}^{-1}(Q')$, then the polynomials defining Q and Q' have a reducible linear combination vanishing on H, and thus cannot contain $\mathrm{Gr}_2(\mathrm{H}^0(\mathcal{F}^*)^*)$. By assumption, each pair

$$\mathrm{H}^0(\mathcal{F}^*), \mathrm{H}^0(\det \mathcal{S}^*_{\mathrm{FX}}), \qquad \mathrm{Gr}_2(\mathrm{H}^0(\mathcal{F}^*)^*), \mathrm{Gr}_2(V), \qquad \mathbb{P}(\textstyle\bigwedge^2 \mathrm{H}^0(\mathcal{F}^*)^*), \mathbb{P}(\bigwedge^2 V)$$

has the same dimension, so also do the subspaces of quadrics containing Grassmannians, making the function (2.14) necessarily surjective. However, there is no fixed point $C \neq \emptyset$ which is contained in the cone of every quadric containing $\mathrm{Gr}_2(V)$, so C must be empty.

Analogously to $\mathcal{F}$, the relative tautological bundle $\mathcal{S}_{\mathbb{L}} \hookrightarrow p^*\mathcal{S}_{\mathrm{FX}}$ induces a diagram (2.13) for $\mathbb{L} = \mathbb{P}\mathcal{S}_{\mathrm{FX}}$. The isomorphism $(f^*)^*$ descends to the Grassmannians as in diagram (2.15) because they are canonically defined as the intersection of all quadrics that contain FX. In turn, this lifts to an isomorphism $\mathbb{P}(\mathrm{H}^0(\mathcal{F}^*)^*) \xrightarrow{\sim} \mathbb{P}(\mathrm{H}^0(\mathcal{S}^*_{\mathrm{FX}})^*)$ because a 1-dimensional subspace is canonically defined as the intersection of all 2-dimensional subspaces containing it. Finally, since this last isomorphism restricts to an isomorphism between each fiber of the projective bundles $\mathbb{P}\mathcal{F}$ and $\mathbb{P}\mathcal{S}_{\mathrm{FX}}$, it lifts to an isomorphism of vector bundles $\mathcal{F} \xrightarrow{\sim} \mathcal{S}_{\mathrm{FX}}$.

$$\begin{tikzcd}
\mathbb{P}\mathcal{F} \arrow[rr, "\varphi_{\mathcal{S}^*_{\mathbb{P}\mathcal{F}}}"] \arrow[dr, "\sim"] \arrow[ddd, two heads] & & \mathbb{P}(\mathrm{H}^0(\mathcal{S}^*_{\mathbb{P}\mathcal{F}})^*) \arrow[r, "\sim"] \arrow[dr, "\sim"] & \mathbb{P}(\mathrm{H}^0(\mathcal{F}^*)^*) \arrow[dr, "\sim"] & \\
& \mathbb{L} = \mathbb{P}\mathcal{S}_{\mathrm{FX}} \arrow[rr, "\varphi_{\mathcal{S}^*_{\mathbb{L}}}"] \arrow[ddl, two heads] & & \mathbb{P}(\mathrm{H}^0(\mathcal{S}^*_{\mathbb{L}})^*) \arrow[r, "\sim"] & \mathbb{P}(\mathrm{H}^0(\mathcal{S}^*_{\mathrm{FX}})^*) \\
& & \mathrm{Gr}_2(\mathrm{H}^0(\mathcal{F}^*)^*) \arrow[r, hook] \arrow[dr, "\sim"] & \mathbb{P}(\mathrm{H}^0(\det \mathcal{S}^*_{\mathbb{P}\mathcal{F}})^*) \arrow[dr, "(f^*)^*", "\sim"'] & \\
\mathrm{FX} \arrow[urr, hook] \arrow[rrr, hook] & & & \mathrm{Gr}_2(\mathrm{H}^0(\mathcal{S}^*_{\mathrm{FX}})^*) \arrow[r, hook] & \mathbb{P}(\mathrm{H}^0(\det \mathcal{S}^*_{\mathrm{FX}})^*)
\end{tikzcd} \tag{2.15}$$

□

Lemma 2.28 *The Plücker embedding $\iota : \mathrm{FX} \hookrightarrow \mathbb{G} \hookrightarrow \mathbb{P}(\bigwedge^2 V)$ is canonical, meaning that*

$$\omega_{\mathrm{FX}} \cong \iota^* \mathcal{S}^*_{\mathbb{P}(\wedge^2 V)} \cong \det \mathcal{S}^*_{\mathrm{FX}} .$$

Proof The tautological sequence $\mathcal{S}_{\mathbb{G}} \hookrightarrow \mathcal{O}_{\mathbb{G}} \otimes V \twoheadrightarrow \mathcal{Q}_{\mathbb{G}}$ on $\mathbb{G}$ shows $\det \mathcal{S}^*_{\mathbb{G}} \cong \det \mathcal{Q}_{\mathbb{G}}$, and it is known that $\mathcal{T}_{\mathbb{G}} \cong \mathcal{S}^*_{\mathbb{G}} \otimes \mathcal{Q}_{\mathbb{G}}$. So we compute

$$\omega_{\mathbb{G}} \cong \det \mathcal{T}^*_{\mathbb{G}} \cong \det(\mathcal{S}_{\mathbb{G}} \otimes \mathcal{Q}^*_{\mathbb{G}}) \cong (\det \mathcal{S}_{\mathbb{G}})^{\otimes 2} \otimes (\det \mathcal{Q}^*_{\mathbb{G}})^{\otimes 3} \cong (\det \mathcal{S}_{\mathbb{G}})^{\otimes 5}.$$

Since the closed subvariety $\mathrm{FX} \hookrightarrow \mathbb{G}$ is cut out by a global section of $\mathcal{O}_{\mathbb{G}}(3)$, we obtain by the adjunction formula

$$\omega_{\mathrm{FX}} \cong (\omega_{\mathbb{G}} \otimes \det(\mathcal{O}_{\mathbb{G}}(3)))|_{\mathrm{FX}} \cong \omega_{\mathbb{G}}|_{\mathrm{FX}} \otimes (\det \mathcal{S}^*_{\mathbb{G}})^{\otimes 6}|_{\mathrm{FX}} \cong (\det \mathcal{S}^*_{\mathbb{G}})|_{\mathrm{FX}} \cong \det \mathcal{S}^*_{\mathrm{FX}}. \qquad \square$$

We will see in Theorem 4.11 that the cotangent bundle $\mathcal{T}^*_{\mathrm{FX}}$ is globally generated and has 5-dimensional global sections, so by Theorem 2.27 it will follow that it is isomorphic to the tautological bundle $\mathcal{S}^*_{\mathrm{FX}}$. It is also possible to prove these facts directly, as in [CG72, §12] through intricate analytical arguments, or in [Tyu70] by bounding the χ_y genus of FX.

2.6 Picard variety of the Fano surface

The Plücker embedding $\iota : \mathrm{F}X \hookrightarrow \mathrm{Gr}(2, V) \hookrightarrow \mathbb{P}(\bigwedge^2 V)$ induces on the Fano surface the integral Kähler form

$$g := \iota^* \mathrm{c}_1(\mathcal{S}^*_{\mathbb{P}(\wedge^2 V)}) = \mathrm{c}_1(\det \mathcal{S}^*_{\mathrm{F}X}) = \mathrm{c}_1(\mathcal{S}^*_{\mathrm{F}X}) \in \mathrm{H}^2(\mathrm{F}X, \mathbb{Z}),$$

which we will also refer to as the Plücker polarization. The Chern classes of $\mathcal{S}_{\mathrm{F}X}$ satisfy by construction

$$\mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}})^2 + \mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}}) \smile p^* \mathrm{c}_1(\mathcal{S}_{\mathrm{F}X}) + p^* \mathrm{c}_2(\mathcal{S}_{\mathrm{F}X}) = 0 \in \mathrm{H}^4(\mathbb{L}, \mathbb{Z}),$$

and the polarizing class $\mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}})$ restricts to a positive generator of $\mathrm{H}^2(\mathbb{L}|_L, \mathbb{Z})$ on each fiber over $L \in \mathrm{F}X$. Thus, evaluation along the fibers p_* satisfies the projection formula

$$\int_{\mathbb{L}} p^*\alpha \smile \mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}}) = \int_{\mathrm{F}X} p_*(p^*\alpha \smile \mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}})) = \int_{\mathrm{F}X} \alpha, \qquad \text{for all } \alpha \in \mathrm{H}^4(\mathrm{F}X, \mathbb{Z}).$$

Moreover, by the Leray-Hirsch theorem, the pullback p^* makes

$$\mathrm{H}^*(\mathbb{L}, \mathbb{Z}) = p^*\mathrm{H}^*(\mathrm{F}X, \mathbb{Z}) \oplus (\mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}}) \smile p^*\mathrm{H}^*(\mathrm{F}X, \mathbb{Z}))$$

a free $\mathrm{H}^*(\mathrm{F}X, \mathbb{Z})$-module.

Lemma 2.29 *Let $L \in \mathrm{F}X \setminus (\mathcal{D}_2 \cup \mathcal{R}_2)$ and $[\mathbb{P}^2] \in \mathrm{H}^4(X, \mathbb{Z})$ the fundamental class of a plane $\mathbb{P}^2 \subset \mathbb{P}V$. Then it holds*

$$g = p_*q^*[\mathbb{P}^2] = 3[C_L] \in \mathrm{H}^2(\mathrm{F}X, \mathbb{Z}).$$

Proof For the first equality, use that $[\mathbb{P}^2] = \mathrm{c}_1(\mathcal{S}^*_X)^2$ is the square of the hyperplane class. With $\mathcal{S}_{\mathbb{L}} = q^*\mathcal{S}_X$, $g = \mathrm{c}_1(\mathcal{S}^*_{\mathrm{F}X})$ and the projection formula for p_* we compute

$$p_*q^* \mathrm{c}_1(\mathcal{S}^*_X)^2 = p_* \mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}})^2 = p_* \left(- \mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}}) \smile p^* \mathrm{c}_1(\mathcal{S}_{\mathrm{F}X}) - \mathrm{c}_2(\mathcal{S}_{\mathrm{F}X})\right) = - \mathrm{c}_1(\mathcal{S}_{\mathrm{F}X}) = g.$$

For the second equality, since the class does not depend on the choice of plane, we can assume that $\mathbb{P}^2 \cap X = L_1 \cup L_2 \cup L_3$ for three (potentially equal) lines. Since the class of a line is also indipendent by Remark 2.19, we obtain

$$[\mathbb{P}^2] = [L_1] + [L_2] + [L_3] = 3[L] \implies p_*q^*[\mathbb{P}^2] = 3p_*q^*[L] = 3[C_L]. \qquad \square$$

On $\mathrm{H}^1(\mathrm{F}X, \underline{\mathbb{C}})$ there is the Hermitian form

$$\mathcal{H}^1_{\mathrm{F}X} : (\alpha, \beta) \mapsto 2i \int_{\mathrm{F}X} \alpha \wedge \overline{\beta} \wedge g,$$

which is non-degenerate when restricted to the right half of the Hodge diamond, and whose imaginary part is integer-valued on the lattice $\rho\,\mathrm{H}^1(\mathrm{F}X,\mathbb{Z})$. As we have seen in Example 1.6, the Hermitian form $-\mathcal{H}^1_{\mathrm{FX}}$ induces a positive polarization on the Picard variety $\mathrm{Pic}^0\,\mathrm{F}X = \widehat{\mathrm{J}^1\,\mathrm{F}X}$, making it an Abelian variety.

Proposition 2.30 *Pulling back by q and integrating along the fibers of p induces a monomorphism*

$$p_*q^* \;:\; \mathrm{H}^{1,2}(X) \hookrightarrow \mathrm{H}^{0,1}(\mathrm{F}X)$$

which pulls back the Hermitian form $-\mathcal{H}^1_{\mathrm{FX}}$ to $6\cdot\mathcal{H}_X$.

Proof The relevant homomorphisms in integral cohomology are

$$\begin{tikzcd}
 & \mathrm{H}^3(\mathbb{L},\mathbb{Z}) \arrow[dl, two heads, "p_*"'] & \mathrm{H}^3(X,\mathbb{Z}) \arrow[l, hook', "q^*"'] \\
\mathrm{H}^1(\mathrm{F}X,\mathbb{Z}) & \mathrm{H}^3(\mathrm{F}X,\mathbb{Z})\oplus\mathrm{H}^1(\mathrm{F}X,\mathbb{Z}) \arrow[l, two heads] \arrow[u, "\wr"', "p^*\oplus(p^*\smile \mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}}))"]
\end{tikzcd}$$

Take $\alpha,\beta \in \mathrm{H}^3(X,\mathbb{Z})$ and $\widetilde{\alpha},\widetilde{\beta} \in \mathrm{H}^3(\mathrm{F}X,\mathbb{Z})$ such that

$$q^*\alpha = p^*\widetilde{\alpha} + \mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}}) \smile p^*p_*q^*\alpha, \quad q^*\beta = p^*\widetilde{\beta} + \mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}}) \smile p^*p_*q^*\beta.$$

Since every cohomology class in $\mathrm{H}^3(X,\mathbb{Z})$ is primitive, it holds

$$\mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}}) \smile q^*\alpha = q^*(\mathrm{c}_1(\mathcal{S}^*_{\mathrm{X}}) \smile \alpha) = 0, \quad \mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}}) \smile q^*\beta = q^*(\mathrm{c}_1(\mathcal{S}^*_{\mathrm{X}}) \smile \beta) = 0.$$

Thus, using $p^*\widetilde{\alpha} \smile p^*\widetilde{\beta} = p^*(\widetilde{\alpha} \smile \widetilde{\beta}) = 0$, it holds also

$$\begin{aligned}
q^*\alpha \smile q^*\beta = q^*\alpha \smile p^*\widetilde{\beta} = \mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}}) \smile p^*(p_*q^*\alpha \smile \widetilde{\beta}),\\
q^*\alpha \smile q^*\beta = q^*\widetilde{\alpha} \smile p^*\beta = \mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}}) \smile p^*(\widetilde{\alpha} \smile p_*q^*\beta).
\end{aligned} \tag{2.16}$$

Naively distributing and applying equations (2.16) we get

$$\begin{aligned}
q^*\alpha \smile q^*\beta &= \mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}}) \smile p^*(\widetilde{\alpha} \smile p_*q^*\beta + p_*q^*\alpha \smile \widetilde{\beta}) + \mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}})^2 \smile p^*p_*q^*(\alpha \smile \beta)\\
&= 2\,q^*\alpha \smile q^*\beta + \mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}})^2 \smile p^*p_*q^*(\alpha \smile \beta),
\end{aligned}$$

which we can use to compute

$$\begin{aligned}
6\int_X \alpha \smile \beta &= \int_{\mathbb{L}} q^*(\alpha \smile \beta) = -\int_{\mathbb{L}} \mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}})^2 \smile p^*p_*q^*(\alpha \smile \beta)\\
&= -\int_{\mathbb{L}} (-\,\mathrm{c}_1(\mathcal{S}^*_{\mathbb{L}}) \smile p^*\,\mathrm{c}_1(\mathcal{S}_{\mathrm{FX}}) - p^*\,\mathrm{c}_2(\mathcal{S}_{\mathrm{FX}})) \smile p^*p_*q^*(\alpha \smile \beta)\\
&= \int_{\mathrm{FX}} \mathrm{c}_1(\mathcal{S}_{\mathrm{FX}}) \smile p_*q^*(\alpha \smile \beta) = -\int_{\mathrm{FX}} p_*q^*\alpha \smile p_*q^*\beta \smile g\,,
\end{aligned}$$

where we have used that $\mathrm{c}_1(\mathcal{S}_{\mathrm{FX}}) = -\,\mathrm{c}_1(\mathcal{S}^*_{\mathrm{FX}}) = -g$. We conclude

$$6\,\Im\mathcal{H}_X(\rho\,\alpha, \rho\,\beta) = -\,\Im\mathcal{H}^1_{\mathrm{FX}}(\rho\; p_*q^*\alpha, \rho\; p_*q^*\beta)\,,$$

and since $\mathcal{H}_X$ is uniquely determined by the values of its imaginary part on the complete lattice $\rho\,\mathrm{H}^3(X,\mathbb{Z})$, we have verified that $-\mathcal{H}^1_{\mathrm{FX}}$ pulls back to $6\mathcal{H}_X$. The resulting homomorphism p_*q^* is injective because $6\mathcal{H}_X$ is non-degenerate. $\square$

Lemma 2.31 *Let $L \in \mathrm{FX} \setminus (\mathcal{D}_2 \cup \mathcal{R}_2)$. Then, the restriction $\iota_L^* : \mathrm{H}^{0,1}(\mathrm{FX}) \hookrightarrow \mathrm{H}^{0,1}(C_L)$ is injective and it pulls back the Hermitian form $\mathcal{H}_{C_L}$ to $\mathcal{H}^1_{\mathrm{FX}}$*

Proof The pull-back of Hermitian forms follows from Lemma 2.29, whereas the injectivity from the non-degeneracy of $\mathcal{H}^1_{\mathrm{FX}}$. $\square$

Now we consider the algebraically trivial Chow groups of the algebraic varieties in diagram (2.8). Morphisms push forward and pull back to group homomorphisms. Since the projections $\overline{\varphi}$, p and $\varphi : \mathrm{Bl}_L X \twoheadrightarrow \mathbb{P}(V/W)$ are flat, the Hilbert polynomial of their fibers being constant 2, $z+1$, and $2z+1$ respectively, their intersection-theoretic pullback is well-defined. Note also that $\mathrm{CH}_k(\mathbb{P}(V/W))_{\mathrm{alg}} = 0$ by definition of algebraic equivalence.

$$\begin{tikzcd}
 & & \mathrm{CH}_1(\mathrm{Bl}_L X)_{\mathrm{alg}} \arrow[d, "\tau_*"] & \\
\mathrm{CH}_1(\mathbb{L})_{\mathrm{alg}} \arrow[rr, "q_*"] & & \mathrm{CH}_1(X)_{\mathrm{alg}} & \mathrm{CH}_0(\mathbb{P}(V/W))_{\mathrm{alg}} = 0 \arrow[ul, "\varphi^*"'] \\
\mathrm{CH}_0(\mathrm{FX})_{\mathrm{alg}} \arrow[u, "p^*"] & \mathrm{CH}_0(C_L)_{\mathrm{alg}} \arrow[l, "\iota_{L*}"'] & \mathrm{CH}_0(D_L)_{\mathrm{alg}} \arrow[l, "\overline{\varphi}^*"'] \arrow[ur] &
\end{tikzcd} \tag{2.17}$$

Lemma 2.32 *The composition $q_*\,p^*\,\iota_{L*}\,\overline{\varphi}^*$ in the following diagram is trivial:*

$$\begin{tikzcd}
\mathrm{H}_1(D_L,\mathbb{Z}) \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{H}_1(C_L,\mathbb{Z}) \arrow[r, "\iota_{L*}"] \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{H}_1(\mathrm{FX},\mathbb{Z}) \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{H}_3(\mathbb{L},\mathbb{Z}) \arrow[r, "q_*"] \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{H}_3(X,\mathbb{Z}) \arrow[d, phantom, "\cong" rotate=-90] \\
\mathrm{H}^1(D_L,\mathbb{Z}) \arrow[r, "\overline{\varphi}^*"] & \mathrm{H}^1(C_L,\mathbb{Z}) & \mathrm{H}^3(\mathrm{FX},\mathbb{Z}) \arrow[r, "p^*"] & \mathrm{H}^3(\mathbb{L},\mathbb{Z}) & \mathrm{H}^3(X,\mathbb{Z}),
\end{tikzcd}$$

where the vertical isomorphisms are Poincaré duality.

Proof The domain and codomain (and actually every group in question) are free of finite rank, so it is enough to prove the claim for the induced complex tori, which are precisely the intermediate Jacobians. The diagram of Chow groups (2.17) maps functorially to the diagram of intermediate Jacobians by the generalized Abel-Jacobi map:

$$\begin{tikzcd}
\mathrm{CH}_0(D_L)_{\mathrm{alg}} \arrow[r, "\overline{\varphi}^*"] \arrow[d, "\Phi^0_{D_L}"', "\wr"] & \mathrm{CH}_0(C_L)_{\mathrm{alg}} \arrow[r, "\iota_{L*}"] \arrow[d, "\Phi^0_{C_L}"', "\wr"] & \mathrm{CH}_0(\mathrm{F}X)_{\mathrm{alg}} \arrow[r, "p^*"] \arrow[d, "\Phi^0_{\mathrm{F}X}"'] & \mathrm{CH}_1(\mathbb{L})_{\mathrm{alg}} \arrow[r, "q_*"] \arrow[d, "\Phi^1_{\mathbb{L}}"'] & \mathrm{CH}_1(X)_{\mathrm{alg}} \arrow[d, "\Phi^1_X"'] \\
\mathrm{J}^1 D_L \arrow[d, phantom, "\cong" rotate=90] & \mathrm{J}^1 C_L \arrow[r, "\iota_{L*}"] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{J}^1 \mathrm{F}X \arrow[d, phantom, "\cong" rotate=90] & \mathrm{J}^3 \mathbb{L} \arrow[r, "q_*"] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{J}^3 X \arrow[d, phantom, "\cong" rotate=90] \\
\widehat{\mathrm{J}^1 D_L} \arrow[r, "\overline{\varphi}^*"] & \widehat{\mathrm{J}^1 C_L} & \widehat{\mathrm{J}^3 \mathrm{F}X} \arrow[r, "p^*"] & \widehat{\mathrm{J}^3 \mathbb{L}} & \widehat{\mathrm{J}^3 X}\,.
\end{tikzcd}$$

The total composition $q_*\, p^*\, \iota_{L*}\, \overline{\varphi}^*$ of the top row vanishes, since it factors through a trivial group in (2.17). Since the Abel-Jacobi maps of divisors on curves are isomorphisms, also the zig zag composition $\mathrm{J}^1 D_L \to \mathrm{J}^3 X$ vanishes. ☐

Corollary 2.33 *The composition $\overline{\varphi}_*\, \iota_L^*$ in the following diagram is trivial:*

$$\begin{tikzcd}
\mathrm{H}^1(D_L,\mathbb{Z}) \arrow[d, phantom, "\cong" rotate=90] & \mathrm{H}^1(C_L,\mathbb{Z}) \arrow[d, phantom, "\cong" rotate=90] & \mathrm{H}^1(\mathrm{F}X,\mathbb{Z}) \arrow[l, "\iota_L^*"'] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{H}^3(\mathbb{L},\mathbb{Z}) \arrow[d, phantom, "\cong" rotate=90] & \mathrm{H}^3(X,\mathbb{Z}) \arrow[l, "q^*"'] \arrow[d, phantom, "\cong" rotate=90] \\
\mathrm{H}_1(D_L,\mathbb{Z}) & \mathrm{H}_1(C_L,\mathbb{Z}) \arrow[l, "\overline{\varphi}_*"'] & \mathrm{H}_3(\mathrm{F}X,\mathbb{Z}) & \mathrm{H}_3(\mathbb{L},\mathbb{Z}) \arrow[l, "p_*"'] & \mathrm{H}_3(X,\mathbb{Z}),
\end{tikzcd}$$

where the vertical isomorphisms are Poincaré duality.

Proof The diagram in question is obtained from the one in Lemma 2.32 by taking the algebraic duals of every group and identifying $\mathrm{H}_k(X,\mathbb{Z})^* \cong \mathrm{H}^k(X,\mathbb{Z})$. Thus, the total composition $\overline{\varphi}_*\, \iota_L^*\, p_*\, q^*$ is trivial. Noting that $p_*\, q^*$ is an isomorphism in complex coeffients, which we proved in Proposition 2.30 by interpreting p_* as integration along the fibers, the claim follows. ☐

Corollary 2.33 shows that the image of the restriction ι_L^* is contained in the Poincaré dual of the kernel of $\overline{\varphi}_*$. We will describe this kernel in Section 3.1 using the involution $\sigma : C_L \xrightarrow{\sim} C_L$ induced by the étale double cover $C_L \twoheadrightarrow D_L$, which sends a line $L' \in C_L$ to its residual in $\langle L, L' \rangle \cap X \subset \mathbb{P}$.

Corollary 2.34 *It holds*

$$\int_{\mathrm{F}X} \alpha \smile \beta \smile g \in 6\mathbb{Z}, \quad \textit{for all } \alpha, \beta \in \mathrm{H}^1(\mathrm{F}X,\mathbb{Z}),$$

making $-\frac{1}{6}\mathcal{H}^1_{\mathrm{F}X}$ a polarization on the complex torus $\mathrm{Pic}^0\,\mathrm{F}X$.

Proof It will suffice to anticipate from the next chapter that the Poincaré dual of $\ker \overline{\varphi}_* \subset \mathrm{H}_1(C_L,\mathbb{Z})$ is $\mathrm{im}(\mathrm{id} - \sigma^*) \subset \mathrm{H}^1(C_L,\mathbb{Z})$, see the exact sequence (3.2). Thus we can write $\alpha|_{C_L} = \widetilde{\alpha} - \sigma^*\widetilde{\alpha}$, $\beta|_{C_L} = \widetilde{\beta} - \sigma^*\widetilde{\beta}$ for $\widetilde{\alpha}, \widetilde{\beta} \in \mathrm{H}^1(C_L,\mathbb{Z})$. So from Lemma 2.31 we know that

$$\int_{\mathrm{FX}} \alpha \smile \beta \smile g = 3\int_{C_L} \alpha|_{C_L} \smile \beta|_{C_L} = 3\left(2\int_{C_L} \widetilde{\alpha} \smile \widetilde{\beta} - 2\int_{C_L} \widetilde{\alpha} \smile \sigma^*\widetilde{\beta}\right) \in 6\mathbb{Z}\,.$$

Theorem 2.35 (Clemens-Griffiths) *There is an isomorphism of Abelian varieties*

$$\left(\widehat{\mathrm{J}^3\,\mathrm{X}}, \mathcal{H}_X\right) \xrightarrow[\sim]{p_*q^*} \left(\mathrm{Pic}^0\,\mathrm{FX}, -\frac{1}{6}\mathcal{H}^1_{\mathrm{FX}}\right)\,.$$

Proof By Proposition 2.30 and Lemma 2.31 we have monomorphisms

$$\mathrm{H}^{1,2}(X) \xhookrightarrow{p_*q^*} \mathrm{H}^{0,1}(\mathrm{FX}) \xhookrightarrow{\iota_L^*} \mathrm{H}^{0,1}(C_L).$$

By Corollary 2.33, the image of ι_L^* is contained in a subspace of dimension $g(C_L) - g(D_L) = 5$, because $\overline{\varphi}_* : \mathrm{H}_1(C_L,\mathbb{R}) \to \mathrm{H}_1(D_L,\mathbb{R})$ is surjective. In integral cohomology, it follows that $p_*q^* : \mathrm{H}^3(X,\mathbb{Z}) \hookrightarrow \mathrm{H}^1(\mathrm{F}X,\mathbb{Z})$ has finite cokernel. Since p_*q^* pulls back the alternating form $-\frac{1}{6}\Im\mathcal{H}^1_{\mathrm{FX}}$ to $\Im\mathcal{H}_X$, which is unimodular, p_*q^* can only be an isomorphism of lattices, and thus of complex tori. □

Chapter 3

Prym Varieties and their Theta Divisors

In this chapter, we present Mumford's theory of Prym varieties [Mum74]. They are Abelian varieties constructed in terms of double covers of curves, and whose geometry is thus accessible in terms of curve theory. We also follow closely [BL04, 12].

Consider an étale double cover $f : C \twoheadrightarrow D$ of smooth, irreducible, projective curves. Let g be the genus of D, and thus $2g-1$ necessarily the genus of C by the Riemann-Hurwitz theorem. Assume moreover $g \geq 2$.

3.1 Cohomology of a double cover

Denote $\sigma : C \xrightarrow{\sim} C$ the involution over D induced by f. The involutions σ_* and σ^* in (co)homology with field coefficients induce eigenspaces decompositions with respect to the eigenvalues ± 1, and come equipped with natural splittings:

$$\begin{array}{ccccc} \mathrm{H}_1(C,\mathbb{R}) & = & \mathrm{H}_1(C,\mathbb{R})^+ \oplus \mathrm{H}_1(C,\mathbb{R})^-, & & \gamma \mapsto (\frac{\gamma+\sigma_*\gamma}{2}, \frac{\gamma-\sigma_*\gamma}{2}), \\ \wr\| & & \wr\| \qquad\quad \wr\| & & \\ \mathrm{H}^1(C,\mathbb{R}) & = & \mathrm{H}^1(C,\mathbb{R})^+ \oplus \mathrm{H}^1(C,\mathbb{R})^-, & & \omega \mapsto (\frac{\omega+\sigma^*\omega}{2}, \frac{\omega-\sigma^*\omega}{2}). \end{array} \tag{3.1}$$

These decompositions are moreover respected by Poincaré duality, because the dual of an inclusion of an eigenspace is the splitting projection onto its dual. In particular, the eigenspaces in integral coefficients $\mathrm{H}_1(C,\mathbb{Z})^\pm := \mathrm{H}_1(C,\mathbb{Z}) \cap \mathrm{H}_1(C,\mathbb{R})^\pm$ are Poincaré dual to $\mathrm{H}^1(C,\mathbb{Z})^\pm := \mathrm{H}^1(C,\mathbb{Z}) \cap \mathrm{H}^1(C,\mathbb{R})^\pm$.

Lemma 3.1 *There exist symplectic bases with respect to the intersection products*

$$\langle \widetilde{\lambda}_1, \widetilde{\mu}_1, (\widetilde{\lambda}_i^+, \widetilde{\lambda}_i^-, \widetilde{\mu}_i^+, \widetilde{\mu}_i^-)_{2\leq i\leq g} \rangle = \mathrm{H}_1(C,\mathbb{Z}), \quad \langle (\lambda_i, \mu_i)_{1\leq i\leq g} \rangle = \mathrm{H}_1(D,\mathbb{Z}),$$

such that $(\widetilde{\lambda}_i^{\pm}\cdot\widetilde{\mu}_j^{\pm})=\delta_{ij}$, $(\lambda_i\cdot\mu_j)=\delta_{ij}$, $\sigma_*\widetilde{\lambda}_1=\widetilde{\lambda}_1$, $\sigma_*\widetilde{\mu}_1=\widetilde{\mu}_1$, $\sigma_*\widetilde{\lambda}_i^{\pm}=\widetilde{\lambda}_i^{\mp}$, $\sigma_*\widetilde{\mu}_i^{\pm}=\widetilde{\mu}_i^{\mp}$,

$$f_*\widetilde{\lambda_1}=2\lambda_1,\quad f_*\widetilde{\mu}_1=\mu_1,\quad f_*\widetilde{\lambda}_i^{+}=f_*\widetilde{\lambda}_i^{-}=\lambda_i,\quad f_*\widetilde{\mu}_i^{+}=f_*\widetilde{\mu}_i^{-}=\mu_i\,.$$

Proof We refer to [BL04, 12.4], where the generators $\widetilde{\lambda}_i^{\pm},\widetilde{\mu}_i^{\pm}$ are constructed from λ_i,μ_i by representing the Riemann surface C topologically as two glued copies of D. $\square$

Corollary 3.2 *For a symplectic basis as in Lemma 3.1, the eigenspace decomposition* (3.1) *in integral homology is*

$$\mathrm{H}_1(C,\mathbb{Z})^+=\langle\widetilde{\lambda}_1,\widetilde{\mu}_1,(\widetilde{\lambda}_i^{+}+\widetilde{\lambda}_i^{-},\widetilde{\mu}_i^{+}+\widetilde{\mu}_i^{-})_{2\le i\le g}\rangle,\ \ \mathrm{H}_1(C,\mathbb{Z})^-=\langle(\widetilde{\lambda}_i^{+}-\widetilde{\lambda}_i^{-},\widetilde{\mu}_i^{+}-\widetilde{\mu}_i^{-})_{2\le i\le g}\rangle,$$

and there are exact sequences

$$\begin{array}{ccccc}
(\mathbb{Z}/2\mathbb{Z})^{\oplus 2g-2} & \twoheadleftarrow & \mathrm{H}_1(C,\mathbb{Z}) & \hookleftarrow & \mathrm{H}_1(C,\mathbb{Z})^+\oplus\mathrm{H}_1(C,\mathbb{Z})^- \\
 & & \wr\| & & \wr\| \qquad \wr\| \\
(\mathbb{Z}/2\mathbb{Z})^{\oplus 2g-2} & \twoheadleftarrow & \mathrm{H}^1(C,\mathbb{Z}) & \hookleftarrow & \mathrm{H}^1(C,\mathbb{Z})^+\oplus\mathrm{H}^1(C,\mathbb{Z})^-,
\end{array}$$

where the cokernels are generated by the images of $(\widetilde{\lambda}_i^{+},\widetilde{\mu}_i^{+})_{2\le i\le g}$.

By definition $\mathrm{H}_1(C,\mathbb{Z})^{\pm}=\ker(\mathrm{id}\mp\sigma_*)$ and $\operatorname{im} f_*\subset\mathrm{H}_1(D,\mathbb{Z})\cong\pi_1(D)$ is the subgroup of cycles that induce trivial monodromy. Using a symplectic basis as in Lemma 3.1 we also see $\operatorname{im}(\mathrm{id}-\sigma_*)=\langle(\widetilde{\lambda}_i^{+}-\widetilde{\lambda}_i^{-},\widetilde{\mu}_i^{+}-\widetilde{\mu}_i^{-})_{2\le i\le g}\rangle=\mathrm{H}_1(C,\mathbb{Z})^-=\ker f_*$. So there is an exact sequence

$$\mathrm{H}_1(C,\mathbb{Z})^+\hookrightarrow\mathrm{H}_1(C,\mathbb{Z})\xrightarrow{\mathrm{id}-\sigma_*}\mathrm{H}_1(C,\mathbb{Z})\xrightarrow{f_*}\mathrm{H}_1(D,\mathbb{Z})\xrightarrow{\text{monodromy}}\mathbb{Z}/2\mathbb{Z}\,.\qquad(3.2)$$

By Poincaré duality, it also holds $\operatorname{im}(\mathrm{id}-\sigma^*)=\operatorname{im}(\mathrm{id}-\sigma_*)^*=\mathrm{H}^1(C,\mathbb{Z})^-=\ker(\mathrm{id}+\sigma^*)$, so there is an exact sequence

$$\mathbb{Z}/2\mathbb{Z}\twoheadleftarrow\mathrm{H}^1(C,\mathbb{Z})^+\xleftarrow{\mathrm{id}+\sigma^*}\mathrm{H}^1(C,\mathbb{Z})\xleftarrow{\mathrm{id}-\sigma^*}\mathrm{H}^1(C,\mathbb{Z})\,,\qquad(3.3)$$

where the 2-torsion generator of $\operatorname{coker}(\mathrm{id}+\sigma^*)$ is the Poincaré dual of $\widetilde{\lambda}_1$. Note that $\ker(\mathrm{id}-\sigma^*)\supset f^*\mathrm{H}^1(D,\mathbb{Z})$ is strictly larger, because the dual of λ_1 is pulled back to twice the dual of $\widetilde{\lambda}_1$.

Applying the Hodge projection to (3.1) we get similar decompositions with respect to the involutions $(\sigma^*)^*$ and σ^*

$$\begin{array}{ccc}
\mathrm{H}^{1,0}(C)^* & = & (\mathrm{H}^{1,0}(C)^*)^+\oplus(\mathrm{H}^{1,0}(C)^*)^- \\
\wr\uparrow\Phi_{-\mathcal{H}_C} & & \wr\uparrow \qquad\qquad \wr\uparrow \\
\mathrm{H}^{0,1}(C) & = & \mathrm{H}^{0,1}(C)^+\oplus\mathrm{H}^{0,1}(C)^-.
\end{array}$$

3.2 The Prym variety of a double cover

Definition 3.3 *The* Prym variety *of the double cover $f : C \twoheadrightarrow D$ is the complex subtorus*

$$\widehat{\mathrm{P}(C/D)} := \mathrm{H}^{0,1}(C)^-/\rho\mathrm{H}^1(C,\mathbb{Z})^- \subset \widehat{\mathrm{J}C}.$$

We denote $\mathrm{P}(C/D)$ its dual complex torus in the sense of Definition A.5.

To describe $\mathrm{P}(C/D)$ we look at the dual of the inclusion $\iota : \mathrm{H}^{0,1}(C)^- \hookrightarrow \mathrm{H}^{0,1}$, which induces a restriction isomorphism

$$\begin{tikzcd}
\mathrm{Hom}_{\overline{\mathbb{C}}}(\mathrm{H}^{0,1}(X)^-, \mathbb{C}) & (\mathrm{H}^{1,0}(C)^*)^- \arrow[l, "\widehat{\iota}"', "\sim"] \arrow[r, hook] & \mathrm{H}^{1,0}(C)^* \\
 & \rho\,\mathrm{H}_1(C,\mathbb{Z})^- \arrow[u, hook] \arrow[r, hook] \arrow[ul, hook] & \rho\,\mathrm{H}_1(C,\mathbb{Z}). \arrow[u, hook]
\end{tikzcd}$$

Passing to complex tori, we obtain a commutative diagram

$$\begin{tikzcd}
\widehat{\mathrm{P}(C/D)} \arrow[r, equal] \arrow[dd, two heads, "\phi_{-\iota^*\mathcal{H}_C}"] & \mathrm{H}^{0,1}(C)^-/\rho\mathrm{H}^1(C,\mathbb{Z})^- \arrow[r, hook, "\iota = \frac{\mathrm{id}-\sigma^*}{2}"] \arrow[d, "\phi_{-\mathcal{H}_C}", "\wr"'] & \widehat{\mathrm{J}C} \arrow[d, "\phi_{-\mathcal{H}_C}", "\wr"'] \\
 & (\mathrm{H}^{1,0}(C)^*)^-/\rho\,\mathrm{H}_1(C,\mathbb{Z})^- \arrow[r, hook] \arrow[dl, two heads, "\widehat{\iota}"] & \mathrm{J}C. \arrow[dll, two heads, bend left, "\widehat{\iota}"] \\
\mathrm{P}(C/D) & &
\end{tikzcd} \tag{3.4}$$

Proposition 3.4 *The Hermitian form $-\frac{1}{2}\mathcal{H}_C$ restricts to a positive-definite, principal polarization on $\widehat{\mathrm{P}(C/D)}$, making it and its dual Abelian varieties.* [1]

Proof By (3.3) it holds $\mathrm{H}^1(C,\mathbb{Z})^- = \mathrm{im}(\mathrm{id} - \sigma^*)$, so we compute

$$\int_C (\alpha - \sigma^*\alpha) \smile (\beta - \sigma^*\beta) = 2\int_C \alpha \smile \beta - 2\int_C \alpha \smile \sigma^*\beta \in 2\mathbb{Z}, \quad \text{for all } \alpha, \beta \in \mathrm{H}^1(C,\mathbb{Z}),$$

where we have used that σ^* preserves the cup product pairing. With the basis elements $\widetilde{\lambda}_i^+ - \widetilde{\lambda}_i^-, \widetilde{\mu}_i^+ - \widetilde{\mu}_i^- \in \mathrm{H}_1(C,\mathbb{Z})^-$ from Lemma 3.1 we verify that the halved cup product pairing is unimodular, so the polarization defined by $-\frac{1}{2}\mathcal{H}_C$ is principal. □

Corollary 3.5 *The restriction $\widehat{\iota}$ induces an exact sequence*

$$(\mathbb{Z}/2\mathbb{Z})^{2g-2} \hookrightarrow (\mathrm{H}^{1,0}(C)^*)^-/\rho\,\mathrm{H}_1(C,\mathbb{Z})^- \twoheadrightarrow \mathrm{P}(C/D)$$

and an isomorphism $\widehat{\iota}$: $(\mathrm{H}^{1,0}(C)^*)^-/\frac{1}{2}\rho\,\mathrm{H}_1(C,\mathbb{Z})^- \xrightarrow{\sim} \mathrm{P}(C/D)$.

[1] Proposition 3.4 means that $(\widehat{\mathrm{P}(C/D)}, -\frac{1}{2}\mathcal{H}_C)$ is a Prym-Tyurin variety for C in the sense of [BL04, 12.2].

As a consequence, the splitting projection $\frac{\mathrm{id}-(\sigma^*)^*}{2}$ with values in $\mathrm{P}(C/D)$ is well defined, and is precisely the restriction $\widehat{\iota}$. It relates to the dualizing isomorphism $\phi_{-\frac{1}{2}\iota^*\mathcal{H}_C}$ through the commutative diagram

$$\begin{tikzcd}
\widehat{\mathrm{P}(C/D)} \arrow[r, equal] \arrow[dd, "\wr"', "\phi_{-\frac{1}{2}\iota^*\mathcal{H}_C}"] & \mathrm{H}^{0,1}(C)^-/\rho\mathrm{H}^1(C,\mathbb{Z})^- \arrow[d, "\wr"', "\phi_{-\mathcal{H}_C}"] & \widehat{\mathrm{J}C} \arrow[l, two heads, "\mathrm{id}-\sigma^*"'] \arrow[d, "\wr"', "\phi_{-\mathcal{H}_C}"] \\
 & (\mathrm{H}^{1,0}(C)^*)^-/\rho\,\mathrm{H}_1(C,\mathbb{Z})^- \arrow[dl, "\cdot\frac{1}{2}", "\sim"'] & \mathrm{J}C\,. \arrow[l, two heads, "\mathrm{id}-(\sigma^*)^*"'] \arrow[dll, two heads, bend left=10, "\widehat{\iota}=\frac{\mathrm{id}-(\sigma^*)^*}{2}"] \\
\mathrm{P}(C/D) & &
\end{tikzcd} \tag{3.5}$$

Dually, the map $\iota' := \widehat{\mathrm{id}-\sigma^*} = \mathrm{id}-(\sigma^*)^*$ realizes $\mathrm{P}(C/D)$ as the complex subtorus $(\mathrm{H}^{1,0}(C)^*)^-/\rho\,\mathrm{H}_1(C,\mathbb{Z})^- \subset \mathrm{J}\,C$ through the commutative diagram

$$\begin{tikzcd}
\widehat{\mathrm{P}(C/D)} \arrow[r, equal] \arrow[dd, "\wr"', "\phi_{-\frac{1}{2}\iota^*\mathcal{H}_C}"] & \mathrm{H}^{0,1}(C)^-/\rho\mathrm{H}^1(C,\mathbb{Z})^- \arrow[r, hook, "\iota"] \arrow[d, "\wr"', "\phi_{-\mathcal{H}_C}"] & \widehat{\mathrm{J}C} \arrow[d, "\wr"', "\phi_{-\mathcal{H}_C}"] \\
 & (\mathrm{H}^{1,0}(C)^*)^-/\rho\,\mathrm{H}_1(C,\mathbb{Z})^- \arrow[r, hook] & \mathrm{J}C\,. \\
\mathrm{P}(C/D) \arrow[ur, "\cdot 2", "\sim"'] \arrow[urr, hook, bend right=10, "\iota':=\widehat{\mathrm{id}-\sigma^*}=\mathrm{id}-(\sigma^*)^*"'] & &
\end{tikzcd} \tag{3.6}$$

Remark 3.6 Corollary 3.5 shows the equality

$$\frac{1}{2}\rho\,\mathrm{H}_1(C,\mathbb{Z})^- = \{\, L \in \mathrm{Hom}_{\overline{\mathbb{C}}}(\mathrm{H}^{0,1}(C)^-,\mathbb{C}) \mid \forall\alpha\in\mathrm{H}^1(C,\mathbb{Z})^- : 2\Re L(\overline{\rho\,\alpha}) \in \mathbb{Z}\},$$

where the right-hand side is by definition the lattice of the dual torus $\mathrm{P}(C/D)$. Indeed, for all $\gamma-\sigma_*\gamma \in \mathrm{im}(\mathrm{id}-\sigma_*) = \mathrm{H}_1(C,\mathbb{Z})^-$, the functional $(\int_{\frac{\gamma-\sigma_*\gamma}{2}} \bullet)$ is in the dual lattice because

$$\int_{\frac{\gamma-\sigma_*\gamma}{2}} \alpha - \sigma^*\alpha = \frac{1}{2}\left(2\int_\gamma \alpha - 2\int_\gamma \sigma^*\alpha\right) \in \mathbb{Z}, \quad \text{for all } \alpha-\sigma^*\alpha \in \mathrm{im}(\mathrm{id}-\sigma^*) = \mathrm{H}^1(C,\mathbb{Z})^-.$$

Moreover, if L is in the dual lattice, then $2\Re L(\overline{\rho\alpha}) = 0$ for all $\alpha \in \mathrm{H}^1(C,\mathbb{Z})^+$, and by writing $L = (\int_\gamma \bullet)$ for some $\gamma \in \mathrm{H}_1(C,\mathbb{R})^-$ it holds

$$\forall\alpha\in\mathrm{H}^1(C,\mathbb{Z})^+\oplus\mathrm{H}^1(C,\mathbb{Z})^- : \int_\gamma \alpha = 2\Re\int_\gamma \overline{\rho\alpha} \in \mathbb{Z}.$$

By Corollary 3.2 it follows that $2\gamma \in \mathrm{H}_1(C,\mathbb{Z})^-$ and $L \in \frac{1}{2}\rho\,\mathrm{H}_1(C,\mathbb{Z})^-$.

Abbreviate by Θ and Ξ the principal polarizations $-\mathcal{H}_C$ on $\widehat{\mathrm{J}C}$ and $-\frac{1}{2}\iota^*\mathcal{H}_C$ on $\widehat{\mathrm{P}(C/D)}$. Let us show how the results above fit into the correspondence between Abelian subvarieties and symmetric idempotents, which draws on the interplay between the dualizing maps ϕ_Θ on $\widehat{\mathrm{J}C}$ and $\phi_{\iota^*\Theta}$ on $\widehat{\mathrm{P}(C/D)}$. It holds $\phi_\Xi^{-1}\phi_{\iota^*\Theta} = 2 \cdot \mathrm{id}$. This can be seen as saying that the inverse ϕ_Ξ^{-1} is the unique isogeny $\mathrm{P}(C/D) \twoheadrightarrow \widehat{\mathrm{P}(C/D)}$ such that $\phi_\Xi^{-1}\phi_{\iota^*\Theta}$ is multiplication by $\exp(\ker(\phi_{\iota^*\Theta}))$, the exponent of the finite group $\ker(\phi_{\iota^*\Theta}) = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2g-2}$. The *norm endomorphism* of the Abelian subvariety $\widehat{\mathrm{P}(C/D)} \subset \widehat{\mathrm{J}C}$ with respect to Θ is then defined as the composition

$$N : \widehat{\mathrm{J}C} \xrightarrow{\phi_\Theta} \mathrm{J}C \xrightarrow{\widehat{\iota}} \mathrm{P}(C/D) \xrightarrow{\phi_\Xi^{-1}} \widehat{\mathrm{P}(C/D)} \xrightarrow{\iota} \widehat{\mathrm{J}C}.$$

By double duality and the identities $\widehat{\phi_\Theta} = \phi_\Theta$, $\widehat{\phi_\Xi^{-1}} = \phi_\Xi^{-1}$, it holds

$$N \circ N = \exp(\ker(\phi_{\iota^*\Theta}))N, \qquad \phi_\Theta^{-1}\,\widehat{N}\,\phi_\Theta = \phi_\Theta^{-1}\,\widehat{\phi_\Theta}\;\iota\;\widehat{\phi_\Xi^{-1}}\;\widehat{\iota}\,\phi_\Theta = \iota\;\phi_\Xi^{-1}\;\widehat{\iota}\,\phi_\Theta = N, \tag{3.7}$$

Viewing N as an element of the endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(\widehat{\mathrm{J}C}) := \mathrm{End}(\widehat{\mathrm{J}C}) \otimes \mathbb{Q}$, the scalar multiple $\frac{1}{2}N$ is idempotent and symmetric with respect to the *Rosati involution* $f \mapsto \phi_\Theta^{-1}\widehat{f}\,\phi_\Theta$ on $\mathrm{End}_{\mathbb{Q}}(\widehat{\mathrm{J}C})$. The correspondence between symmetric idempotents and Abelian subvarieties [BL04, 5.3] then says that N is the unique endomorphism such that (3.7) holds and $\mathrm{im}(N) = \widehat{\mathrm{P}(C/D)}$. Indeed, from diagram (3.5) we see that $N = \mathrm{id} - \sigma^*$, which satisfies

$$(\mathrm{id} - \sigma^*) \circ (\mathrm{id} - \sigma^*) = 2(\mathrm{id} - \sigma^*), \quad \phi_\Theta^{-1}\,\widehat{(\mathrm{id} - \sigma^*)}\,\phi_\Theta = \phi_\Theta^{-1}(\mathrm{id} - (\sigma^*)^*)\,\phi_\Theta = (\mathrm{id} - \sigma^*)$$

and $\mathrm{im}(\mathrm{id} - \sigma^*) = \widehat{\mathrm{P}(C/D)}$.

Analogously, the norm endomorphism of the Abelian subvariety $\mathrm{P}(C/D) \xhookrightarrow{\iota'} \mathrm{J}C$ with respect to the dual polarization of Θ is the composition

$$N' : \mathrm{J}C \xrightarrow{\phi_\Theta^{-1}} \widehat{\mathrm{J}C} \xrightarrow{\widehat{\iota'}} \widehat{\mathrm{P}(C/D)} \xrightarrow{\phi_\Xi} \mathrm{P}(C/D) \xrightarrow{\iota'} \mathrm{J}C,$$

which, again by diagram (3.5), is equal to $\mathrm{id} - (\sigma^*)^*$.

Lemma 3.7 *The Prym variety* $\mathrm{P}(C/D)$ *is a subgroup of index* 2 *in* $\ker f_*$.

Proof On the level of vector spaces it holds $\ker(\mathrm{id} - \sigma^*) = \mathrm{H}^{0,1}(C)^+ = \mathrm{im}\, f^*$, so also after passing to complex tori. Moreover, $\ker\left(f^* : \widehat{\mathrm{J}D} \to \widehat{\mathrm{J}C}\right) \cong \langle \frac{\widetilde{\lambda}_1}{2} \rangle \cong \mathbb{Z}/2\mathbb{Z}$, so The diagram of groups on the left-hand side of (3.8) has exact sequences.

Taking dual complex tori, we obtain the diagram on the right-hand side. The horizontal row then stays exact at $\mathrm{J}C$, because its dual was an exact sequence of complex tori. Moreover, recall that ι' is the dual of $\mathrm{id} - \sigma^*$, and f_* is the dual of f^*. There is an induced map

$\ker f_* \to \mathbb{Z}/2\mathbb{Z}$, which is surjective by the five lemma, and whose kernel is isomorphic to the kernel of $\mathrm{J}\,C \twoheadrightarrow \widehat{\operatorname{im} f^*}$.

$$
\begin{tikzcd}
\widehat{\mathrm{P}(C/D)} & \widehat{\mathrm{J}\,C} \arrow[l, two heads, "\mathrm{id} - \sigma^*"'] & \operatorname{im} f^* \arrow[l, hook'] \\
& \widehat{\mathrm{J}\,D} \arrow[u, "f^*"] \arrow[r, equal] & \widehat{\mathrm{J}\,D} \arrow[u, two heads, "f^*"] \\
& \mathbb{Z}/2\mathbb{Z} \arrow[u, hook] & \mathbb{Z}/2\mathbb{Z} \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \ker f_* \arrow[r, two heads] \arrow[d, hook] & \mathbb{Z}/2\mathbb{Z} \arrow[d, hook] \\
\mathrm{P}(C/D) \arrow[r, hook, "\iota'"] & \mathrm{J}\,C \arrow[r, two heads] \arrow[d, two heads, "f_* = \widehat{f^*}"'] & \widehat{\operatorname{im} f^*} \arrow[d, two heads] \\
& \mathrm{J}\,D \arrow[r, equal] & \mathrm{J}\,D
\end{tikzcd}
\tag{3.8}
$$

□

3.3 Singularities of the theta divisor

We want to study the geometry of a theta divisor for $\widehat{\mathrm{P}(C/D)}$ in terms of Brill-Noether theory. To do this, we start to make use of the moduli interpretation of the Jacobian as the Picard group of line bundles of degree 0. The pushforward f_* of divisors induces the *norm map* $N_f : \operatorname{Pic} C \to \operatorname{Pic} D$ on line bundles, which in degree 0 simply extends the pushforward on Jacobians: [2]

$$
\begin{tikzcd}
& \operatorname{Pic}^0 C \arrow[r, "N_f"] & \operatorname{Pic}^0 D \\
\widehat{\mathrm{P}(C/D)} \arrow[r, hook, "\iota"] \arrow[d, "\wr"', "\phi_{-\frac{1}{2}\iota^*\mathcal{H}_C}"] & \widehat{\mathrm{J}\,C} \arrow[u, "\wr"] \arrow[d, "\wr"', "\phi_{-\mathcal{H}_C}"] & \widehat{\mathrm{J}\,D} \arrow[u, "\wr"] \arrow[d, "\wr"', "\phi_{-\mathcal{H}_D}"] \\
\mathrm{P}(C/D) \arrow[r, hook, "\iota'"] & \mathrm{J}\,C \arrow[r, two heads, "f_*"] & \mathrm{J}\,D
\end{tikzcd}
\tag{3.9}
$$

Proposition 3.8 *The decomposition of* $\ker N_f \subset \operatorname{Pic}^0 C$ *in connected components is*

$$\{\mathcal{F} \otimes \sigma^*\mathcal{F}^{-1} \mid \mathcal{F} \in \operatorname{Pic} C,\ \deg \mathcal{F} \equiv 0 \bmod 2\} \sqcup \{\mathcal{F} \otimes \sigma^*\mathcal{F}^{-1} \mid \mathcal{F} \in \operatorname{Pic} C,\ \deg \mathcal{F} \equiv 1 \bmod 2\},$$

where the first is the Prym variety $\widehat{\mathrm{P}(C/D)} \subset \ker N_f$ *and the second is its unique coset.*

Proof Observe that the first set is a subgroup, and the second is a coset of it. For disjointness, let $\mathcal{F}, \mathcal{E} \in \operatorname{Pic} C$ with $\mathcal{F} \otimes \sigma^*\mathcal{F}^{-1} \cong \mathcal{E} \otimes \sigma^*\mathcal{E}^{-1}$, which is equivalent to

[2] We emphasize not to confuse the norm map N_f with the pushforward of line bundles $\mathcal{L} \in \operatorname{Pic} C \mapsto \det f_*\mathcal{L} \in \operatorname{Pic} D$. For every divisor $E \in \mathrm{CH}_0(C)$, they are related by $\det f_*(\mathcal{O}_C(E)) \cong \mathcal{O}_D(f_*E) \otimes \det f_*\mathcal{O}_C \in \operatorname{Pic} D$.

$(\mathcal{F} \otimes \mathcal{E}^{-1}) \cong \sigma^*(\mathcal{F} \otimes \mathcal{E}^{-1})$. Then, there exists an $\mathcal{M} \in \operatorname{Pic} D$ such that $\mathcal{F} \otimes \mathcal{E}^{-1} \cong f^*\mathcal{M}$, which has $\deg f^*\mathcal{M} = 2 \deg \mathcal{M}$. So $\deg \mathcal{F}$ and $\deg \mathcal{E}$ must have the same parity.

We have already seen in (3.5) that $\widehat{\mathrm{P}(C/D)} \subset \ker N_f$ is the image of the map $\mathcal{F} \in \operatorname{Pic}^0 C \mapsto \mathcal{F} \otimes \sigma^*\mathcal{F}^{-1} \in \operatorname{Pic}^0 C$, so it is contained in the first set of the decomposition. By Lemma 3.7, the index of $\widehat{\mathrm{P}(C/D)} \subset \ker N_f$ is 2, so it has a unique other coset. Then, the Prym variety must already be the whole first set. Moreover, the image of $\widehat{\mathrm{J}C}$ under $\mathrm{id} - \sigma^*$ and the translate of it by a group element are connected. □

Let us recall Riemann's theorem on the divisor $\Theta \subset \mathrm{J}\,D$ of the classical theta function, which is a section of the characteristic 0 line bundle with polarizing class $-\mathcal{H}_D$; we refer to [BL04, 11.2] for a robust treatment. For every $\kappa \in \operatorname{Pic}^{g-1} D$ there is an isomorphism

$$\alpha_\kappa : \operatorname{Pic}^{g-1} D \xrightarrow[\sim]{\mathcal{L} \mapsto \mathcal{L} \otimes \kappa^{-1}} \operatorname{Pic}^0 D \cong \widehat{\mathrm{J}D}\,.$$

In modern terminology, Riemann's theorem states that there exists a $\kappa \in \operatorname{Pic}^{g-1} D$ such that

$$W_{g-1}(D) := \{\mathcal{L} \in \operatorname{Pic}^{g-1} D \mid \mathrm{h}^0(\mathcal{L}) > 0\} = \{\mathcal{L} \otimes \kappa \mid \mathcal{L} \in \Theta\} = \alpha_\kappa^{-1}(\Theta).$$

Then, by Riemann-Roch and Serre duality, $\Theta \subset \operatorname{Pic}^0 D$ is invariant under the involution $\mathcal{L} \mapsto \mathcal{L}^{-1} \otimes \kappa^{-2} \otimes \omega_D$. Since the classical theta function is even, meaning symmetric with respect multiplication by -1, Θ is also invariant under $\mathcal{L} \mapsto \mathcal{L}^{-1}$. Combining the two involutions, we obtain that Θ is invariant under translation by $\kappa^{-2} \otimes \omega_D$. Since Θ defines a principal polarization, the dualizing map ϕ_Θ is an isomorphism; it is described in diagram (3.10), making use of the moduli interpretation of the dual Abelian variety of $\widehat{\mathrm{J}D}$, i.e. $\mathrm{J}\,D$, as the Picard group of $\widehat{\mathrm{J}D}$. Thus, the translation invariance $t^*_{\kappa^{-2}\otimes\omega_D}\Theta = \Theta$ implies $\kappa^2 \cong \omega_D$.

$$\begin{tikzcd}
\widehat{\mathrm{J}D} \arrow[rr, "\phi_\Theta", "\sim"'] \arrow[d, "\wr"] & & \mathrm{J}\,D \arrow[d, "\wr"] \\
\operatorname{Pic}^0 D \arrow[rr, "\mathcal{L} \mapsto \mathcal{O}(t^*_{\mathcal{L}}\Theta) \otimes \mathcal{O}(\Theta)^{-1}", "\sim"'] & & \operatorname{Pic}^0 \widehat{\mathrm{J}D}\,.
\end{tikzcd} \tag{3.10}$$

Line bundles $\kappa \in \operatorname{Pic}^{g-1} D$ with $\kappa^2 \cong \omega_D$ are called *theta characteristics*, and by choosing any one of them, the map α_κ induces an isomorphism

$$\{\chi \in \operatorname{Pic}^{g-1} D \mid \chi^2 \cong \omega_D\} \xrightarrow[\sim]{\alpha_\kappa} \{\eta \in \operatorname{Pic}^0 D \mid \eta^2 \cong \mathcal{O}_D\} \cong (\mathbb{Z}/2\mathbb{Z})^{2g}\,.$$

Again since the classical theta function is even, the multiplicity of Θ at $\mathcal{O}_D$ must be even. By Riemann's theorem on the singularities of Θ, it holds

$$\mathrm{h}^0(\kappa) = \operatorname{mult}_\kappa W_{g-1}(D) = \operatorname{mult}_{\mathcal{O}_D} \Theta \equiv 0 \mod 2\,.$$

More generally, the subset of all theta characteristics κ such that $\mathrm{h}^0(\kappa) \equiv 0 \bmod 2$ can be determined to have size $2^{g-1}(2^g+1)$. This is done by computing the dimension of the eigenspace of all theta functions with respect to the involution $\theta(\bullet) \mapsto \theta(-\bullet)$; see [BL04, 4.7.7].

Lemma 3.9 *There exists a theta characteristic κ' on D such that the theta characteristic $f^*\kappa'$ on C satisfies* $\mathrm{h}^0(f^*\kappa') \equiv 0 \bmod 2$.

Proof The étale double cover f induces a 2-torsion element $\eta \in \operatorname{Pic}^0 D$ such that $f_*f^*\mathcal{L} \cong \mathcal{L} \oplus \mathcal{L} \otimes \eta$, for all $\mathcal{L} \in \operatorname{Pic}^0 D$; see for example [Har77, Ex. **II**.2.7]. So we want to enforce

$$\mathrm{h}^0(f^*\kappa') = \mathrm{h}^0(f_*f^*\kappa') = \mathrm{h}^0(\kappa') + \mathrm{h}^0(\kappa' \otimes \eta) \equiv 0 \mod 2\,. \tag{3.11}$$

Tensoring by η induces an involution on the set of theta characteristics. So the amount $2^{g-1}(2^g+1)$ of theta characteristics κ' with $\mathrm{h}^0(\kappa') \equiv 0 \bmod 2$ equals to the one of theta characteristics with $\mathrm{h}^0(\kappa' \otimes \eta) \equiv 0 \bmod 2$. Since in total there are $2^{2g} < 2 \cdot 2^{g-1}(2^g+1)$ theta characteristics, there exists at least one which enforces (3.11). ☐

Let κ' and $\kappa := f^*\kappa'$ be theta characteristics with $\mathrm{h}^0(\kappa) \equiv 0 \bmod 2$ for the rest of this section.

Proposition 3.10 *The decomposition into connected components of Proposition 3.8 is*

$$\ker N_f = \{\, \mathcal{L} \in \ker N_f \mid \mathrm{h}^0(\mathcal{L} \otimes \kappa) \equiv 0 \mod 2\} \sqcup \{\, \mathcal{L} \in \ker N_f \mid \mathrm{h}^0(\mathcal{L} \otimes \kappa) \equiv 1 \mod 2\}.$$

Proof **Step I:** The function $\mathcal{L} \in \ker N_f \mapsto \mathrm{h}^0(\mathcal{L} \otimes \kappa) \bmod 2$ is constant on connected components. The proof relies on a theorem of Mumford on families of vector bundles with a quadratic form. A theta characteristic κ' enables one to construct a non-degenerate quadratic form on the vector bundles $f_*(\mathcal{L} \otimes \kappa)$ with values in ω_D, varying in a family of all $\mathcal{L} \in \ker N_f$ over D. Mumford's theorem [Mum71] then implies that $\mathrm{h}^0(f_*(\mathcal{L} \otimes \kappa))$ mod 2 is constant on connected components. For the details we refer to [BL04, 12.6].

Step II: The trivial bundle lies in the first component of Proposition 3.8 and satisfies $\mathrm{h}^0(\mathcal{O}_C \otimes \kappa) \equiv 0 \bmod 2$, so this holds on the whole component.

Step III: It remains to establish the existence of an $\mathcal{L} \in \ker N_f$ with $\mathrm{h}^0(\mathcal{L} \otimes \kappa) \equiv 1$ mod 2, which then holds on the whole second component. First, we make sure there is an $\mathcal{L} \in \ker N_f$ with $\mathrm{h}^0(\mathcal{L} \otimes \kappa) > 0$. The norm map is surjective, so we can take an $\mathcal{E} \in \operatorname{Pic} C$ with $N_f\mathcal{E} = \omega_D$. Then, we have $\mathrm{h}^0(\mathcal{E}) \geq \mathrm{h}^0(N_f\mathcal{E}) = g > 0$, and $\mathcal{E} \otimes \kappa^{-1} \in \ker N_f$ because

$$N_f(\mathcal{E} \otimes \kappa^{-1}) \cong \omega_D \otimes N_f\, f^*\kappa'^{-1} \cong \omega_D \otimes \kappa'^{-2} \cong \mathcal{O}_D.$$

Now we show that for every $\mathcal{L} \in \ker N_f$ with $h^0(\mathcal{L} \otimes \kappa) > 0$, there exists a $p \in C$ such that

$$h^0(\mathcal{L}(p - \sigma p) \otimes \kappa) = h^0(\mathcal{L} \otimes \kappa) - 1\,,$$

so line bundles of both parities must exist in $\ker N_f$. Since it also holds $h^0(\mathcal{L}^{-1} \otimes \kappa) = h^1(\mathcal{L}^{-1} \otimes \kappa) = h^0(\mathcal{L} \otimes \kappa) > 0$ by Riemann-Roch, we have non-empty linear systems $|\mathcal{L} \otimes \kappa|, |\mathcal{L}^{-1} \otimes \kappa|$. So we can choose p outside the base locus of $|\mathcal{L}^{-1} \otimes \kappa|$, and such that σp is outside the base locus of $|\mathcal{L} \otimes \kappa|$. In particular, σp is outside the base locus of $|\mathcal{L}(p) \otimes \kappa|$, and it follows

$$h^0(\mathcal{L}(p - \sigma p) \otimes \kappa) = h^0(\mathcal{L}(p) \otimes \kappa) - 1 = h^0(\mathcal{L}^{-1}(-p) \otimes \kappa) = h^0(\mathcal{L}^{-1} \otimes \kappa) - 1. \qquad \square$$

By Proposition 3.4, for any theta divisor Ξ for $(\widehat{\mathrm{P}(C/D)}, -\frac{1}{2}\mathcal{H}_C)$ there is an equality $\iota^*[\Theta] = 2[\Xi]$ of cohomology classes. The next result shows that Ξ can be chosen so that its geometry as an actual subscheme of $\widehat{\mathrm{P}(C/D)}$ is accessible in terms of a theta divisor for $\widehat{\mathrm{J}C}$.

Lemma 3.11 *Consider the theta divisor $\Theta := \alpha_\kappa(W_{2g-2}(C))$ for $(\widehat{\mathrm{J}C}, -\mathcal{H}_C)$. The reduced part of the pulled back divisor on $\widehat{\mathrm{P}(C/D)}$ satisfies $2(\iota^*\Theta)_{\mathrm{red}} = \iota^*\Theta$, as subschemes. In particular, $(\iota^*\Theta)_{\mathrm{red}}$ is a theta divisor for $(\widehat{\mathrm{P}(C/D)} - \frac{1}{2}\mathcal{H}_C)$.*

Proof By Proposition 3.10, we have $\operatorname{mult}_{\mathcal{L}} \Theta = h^0(\mathcal{L} \otimes \kappa) \geq 2$ for all $\mathcal{L} \in \Theta \cap \widehat{\mathrm{P}(C/D)}$. So it holds $2(\iota^*\Theta)_{\mathrm{red}} \leq \Theta$ as effective divisors. On the other hand, if Ξ is any theta divisor for $\widehat{\mathrm{P}(C/D)}$ defining the principal polarization, then $\iota^*[\Theta] = 2[\Xi]$ implies that

$$\exists\, \mathcal{L} \in \operatorname{Pic}^0 C : \quad \mathcal{O}(\Theta) \otimes \mathcal{O}(2\Xi)^{-1} \cong \mathcal{O}(2t^*_{\mathcal{L}}\Xi) \otimes \mathcal{O}(2\Xi)^{-1},$$

meaning that $\Theta \sim_{\mathrm{rat}} 2t^*_{\mathcal{L}}\Xi$. Since $t^*_{\mathcal{L}}\Xi$ is also principal, $(\iota^*\Theta)_{\mathrm{red}}$ must be the unique effective divisor in the linear system $|t^*_{\mathcal{L}}\Xi|$. $\square$

Consider from now on the theta divisors $\Theta := \alpha_\kappa(W_{2g-2}(C))$ and $\Xi := (\iota^*\Theta)_{\mathrm{red}}$. We denote by $\mathrm{TC}_{\mathcal{L}}\,\Theta$ the tangent cone of Θ at $\mathcal{L}$.

Corollary 3.12

$$\operatorname{sing}\Xi = \left\{ \mathcal{L} \in \widehat{\mathrm{P}(C/D)} \,\middle|\, \operatorname{mult}_{\mathcal{L}} \Theta \geq 4 \right\} \sqcup \left\{ \mathcal{L} \in \widehat{\mathrm{P}(C/D)} \,\middle|\, {\scriptstyle \operatorname{mult}_{\mathcal{L}} \Theta = 2 \atop \mathrm{T}_{\mathcal{L}} \widehat{\mathrm{P}(C/D)} \subseteq \mathrm{TC}_{\mathcal{L}} \Theta} \right\}$$

Proof By construction $\Xi \subseteq \operatorname{sing}\Theta$, and if θ is a locally defining holomorphic function for Θ at $\mathcal{L} \in \Xi$, then

$$\operatorname{mult}_{\mathcal{L}} \Xi = \frac{1}{2} \operatorname{ord}_{\mathcal{L}} \theta_{|Z} \geq \frac{1}{2} \operatorname{ord}_{\mathcal{L}} \theta = \frac{1}{2} \operatorname{mult}_{\mathcal{L}} \Theta\,.$$

Inequality holds if and only if the lowest degree terms of θ vanish when restricted to $\mathrm{P}\widehat{(C/D)}$, which means precisely $\mathrm{T}_{\mathcal{L}}\,\mathrm{P}\widehat{(C/D)} \subseteq \mathrm{TC}_{\mathcal{L}}\,\Theta$. $\square$

A line bundle $\mathcal{L} \in \operatorname{sing}\Xi$ is called *exceptional* if $\mathrm{mult}_{\mathcal{L}}\,\Theta = 2$. Using Kempf's characterization of the tangent cone at singularities of Θ, we will now obtain a more explicit description of exceptional singularities of Ξ in terms of line bundles on C and D.

Proposition 3.13

$$\left\{\mathcal{L} \in \mathrm{P}\widehat{(C/D)} \;\middle|\; \substack{\mathrm{mult}_{\mathcal{L}}\,\Theta=2 \\ \mathrm{T}_{\mathcal{L}}\,\mathrm{P}\widehat{(C/D)}\subseteq \mathrm{TC}_{\mathcal{L}}\,\Theta}\right\} = \left\{ f^*\mathcal{M}\otimes\mathcal{N}\otimes\kappa^{-1} \;\middle|\; \substack{\mathcal{M}\in\mathrm{Pic}\,D,\, \mathcal{N}\in\mathrm{Pic}\,C,\; \mathrm{h}^0(\mathcal{N})\geq 1, \\ \mathrm{h}^0(\mathcal{M})=\mathrm{h}^0(f^*\mathcal{M}\otimes\mathcal{N})=2, \\ \mathcal{M}\text{ basepoint-free},\; \mathcal{M}^2\otimes N_f\mathcal{N}\cong\omega_D}\right\}$$

Proof Let $\mathcal{L} \in \operatorname{sing}\Xi$ with $\mathrm{mult}_{\mathcal{L}}\,\Theta = \mathrm{h}^0(\mathcal{L}\otimes\kappa) = 2$. The tangent cone turns out to be more naturally described in the covariant setting, treating the theta divisor as a subset of $\mathrm{J}C$. We need to consider the tangent space $\mathrm{T}_{\mathcal{L}}\,\mathrm{J}C \cong \mathrm{H}^0(\omega_C)^*$ as the affine scheme $\mathrm{Spec}(\mathrm{S}^\bullet\,\mathrm{H}^0(\omega_C))$, where $\mathrm{S}^\bullet\,\mathrm{H}^0(\omega_C)$ is the ring of polynomial functions on $\mathrm{H}^0(\omega_C)^*$ by double duality. The closed subscheme $\mathrm{T}_{\mathcal{L}}\,\mathrm{P}(C/D)$ is then the vanishing locus of the linear subspace $\mathrm{im}(\mathrm{id}+\sigma^*) \subset \mathrm{S}^\bullet\,\mathrm{H}^0(\omega_C)$. Since $\mathcal{L} \in \ker N_f$, we have

$$\mathcal{L}\otimes\sigma^*\mathcal{L} \xrightarrow{\sim} f^*N_f\mathcal{L} \cong \mathcal{O}_C, \quad (\mathcal{L}\otimes\kappa)\otimes(\sigma^*\mathcal{L}\otimes\kappa) \xrightarrow{\sim} \omega_C\,, \tag{3.12}$$

so we can fix vector space bases

$$\mathrm{H}^0(\mathcal{L}\otimes\kappa) = \langle s_1, s_2\rangle, \quad \mathrm{H}^0(\mathcal{L}^{-1}\otimes\kappa) \cong \mathrm{H}^0(\sigma^*\mathcal{L}\otimes\kappa) = \langle \sigma^*s_1, \sigma^*s_2\rangle\,.$$

Kempf's theorem [ACGH85, IV.2] says that the closed subscheme $\mathrm{TC}_{\mathcal{L}}\,\Theta \subset \mathrm{T}_{\mathcal{L}}\,\mathrm{J}C$ is the vanishing locus of the polynomial $\det(s_i\otimes\sigma^*s_j)_{ij} \in \mathrm{S}^2\,\mathrm{H}^0(\omega_C)$, where we treat the products $s_i\otimes\sigma^*s_j$ as sections in $\mathrm{H}^0(\omega_C)$ through (3.12). Then, $\mathrm{T}_{\mathcal{L}}\,\mathrm{P}(C/D)$ factors through $\mathrm{TC}_{\mathcal{L}}\,\Theta$ as a closed subscheme if and only if $\det(s_i\otimes\sigma^*s_j)_{ij}$ is contained in the ideal generated by $\mathrm{im}(\mathrm{id}+\sigma^*)$.

$$\begin{tikzcd}
\mathrm{T}_{\mathcal{L}}\,\mathrm{P}(C/D) \arrow[r, hook] \arrow[dr, dashed] & \mathrm{T}_{\mathcal{L}}\,\mathrm{J}C & & (\mathrm{im}(\mathrm{id}+\sigma^*)) \arrow[r] & \mathrm{S}^\bullet\,\mathrm{H}^0(\omega_C) \\
& \mathrm{TC}_{\mathcal{L}}\,\Theta \arrow[u, hook] & \Longleftrightarrow & & (\det(s_i\otimes\sigma^*s_j)_{ij}) \arrow[u, hook] \arrow[ul, dashed]
\end{tikzcd}$$

We compute $\det(s_i\otimes\sigma^*s_j)_{ij} = s_1\otimes\sigma^*s_1 \cdot s_2\otimes\sigma^*s_2 \,-\, s_1\otimes\sigma^*s_2 \cdot s_2\otimes\sigma^*s_1$, where $s_i\otimes\sigma^*s_i$ are σ^*-invariant and thus contained in $\mathrm{im}(\mathrm{id}+\sigma^*)$. Since $(\mathrm{im}(\mathrm{id}+\sigma^*))$ is a prime ideal, letting $t := s_1\otimes\sigma^*s_2$, we deduce

$$\begin{aligned}\det(s_i\otimes\sigma^*s_j)_{ij} \in (\mathrm{im}(\mathrm{id}+\sigma^*)) \iff t\cdot\sigma^*t \in (\mathrm{im}(\mathrm{id}+\sigma^*)) \iff t \in \mathrm{im}(\mathrm{id}+\sigma^*) \\ \iff t = \sigma^*t \iff s_1\otimes\sigma^*s_2 = s_2\otimes\sigma^*s_1\,.\end{aligned} \tag{3.13}$$

Denote $B \subset C$ the base locus of $|\mathcal{L} \otimes \kappa|$, and consider s_1, s_2 as sections of $\mathcal{L} \otimes \kappa(-B)$. Then

$$\text{(3.13)} \iff \forall p \in C: [s_1|_p : s_2|_p] = [\sigma^* s_1|_p : \sigma^* s_2|_p] = [s_1|_{\sigma p} : s_2|_{\sigma p}] \in \mathbb{P}^1$$
$$\iff \psi : p \in C \mapsto [s_1|_p : s_2|_p] \in \mathbb{P}^1 \text{ is } \sigma\text{-invariant} \iff \psi \text{ factors through } f. \quad (3.14)$$

Moreover, the meromorphic function ψ is the composition of the induced projective morphism $\varphi_{\mathcal{L}\otimes\kappa(-B)}$ with the evaluations on s_1,s_2:

$$\begin{tikzcd}
C \arrow[r, "\varphi_{\mathcal{L}\otimes\kappa(-B)}"] \arrow[d, two heads, "f"'] \arrow[dr, "\psi"] & \mathbb{P}\,\mathrm{H}^0(\mathcal{L} \otimes \kappa(-B))^* \arrow[d, "\wr"'] & {[\lambda]} \arrow[d, mapsto] \\
D \arrow[r, dashed] & \mathbb{P}^1 & {[\lambda(s_1) : \lambda(s_2)]}
\end{tikzcd}$$

So if (3.14) holds, there is an $\mathcal{M} \in \operatorname{Pic} D$ with $\mathcal{L} \otimes \kappa(-B) \cong f^*\mathcal{M}$, which then satisfies

$$2 \leq \mathrm{h}^0(\mathcal{O}_{\mathbb{P}^1}(1)) \leq \mathrm{h}^0(\mathcal{M}) \leq \mathrm{h}^0(\mathcal{L} \otimes \kappa(-B)) = \mathrm{h}^0(\mathcal{L} \otimes \kappa) = 2.$$

For the converse inclusion, one verifies that the above argument can be run backwards using the induced projective morphism $\varphi_{f^*\mathcal{M}} : C \to \mathbb{P}\,\mathrm{H}^0(f^*\mathcal{M})^* \cong \mathbb{P}\,\mathrm{H}^0(f^*\mathcal{M} \otimes \mathcal{N})^*$, which factors through f to $\varphi_{\mathcal{M}} : D \to \mathbb{P}\,\mathrm{H}^0(\mathcal{M})^* \cong \mathbb{P}\,\mathrm{H}^0(f^*\mathcal{M})^*$, and whose global sections thus satisfy (3.14). $\square$

Chapter 4

Intermediate Jacobian of the Cubic Threefold

Let $X \subset \mathbb{P}V$ be a smooth cubic threefold as in Chapter 2, and $L = \mathbb{P}W \in \mathrm{F}X \setminus (\mathcal{D}_2 \cup \mathcal{R}_2)$ a line of first type which does not intersect any line of second type. We will apply the theory of Prym varieties to the étale double cover $\overline{\varphi} : C_L \twoheadrightarrow D_L$ of Section 2.4. After proving that the intermediate Jacobian $\mathrm{J}^3 X$ is isomorphic to the Prym variety $\mathrm{P}(C_L/D_L)$, we will be able to study its theta divisor in terms of curve theory of C_L and D_L.

4.1 Line bundles on C_L and D_L

We gather some results on linear systems of divisors on the curves C_L and D_L. Recall that they are of genus 11 and 6 respectively, and they admit an étale double cover $\overline{\varphi}$ which fits in the following localization of diagram (2.8):

$$\begin{tikzcd}
& \mathrm{Bl}_L X \arrow[d, two heads, "\tau"] \arrow[dr, two heads, "\varphi"] & \\
L \arrow[r, hook] & X & \mathbb{P}(V/W) \\
C_L \arrow[u, two heads, "\deg 5", "q"'] \arrow[r, two heads, "\overline{\varphi}", "\deg 2"'] & D_L \arrow[ur, hook] &
\end{tikzcd}$$

At all points $x \in L$, the embedded tangent hyperplane $\mathbb{T}_x X \subset \mathbb{P}V$ contains every line in X passing through x. The strict transform of $\mathbb{T}_x X$ in $\mathrm{Bl}_L X$ then maps to a line $\varphi(\mathbb{T}_x X) \subset \mathbb{P}(V/W)$, which intersects D_L in all points a whose corresponding plane $\tau(\varphi^{-1}(a)) \subset \mathbb{P}V$ contains a line in X passing through x and different from L. The lines through x different from L are $q^{-1}(x) \subset C_L$, which maps to $\overline{\varphi}(q^{-1}(x)) = D_L \cap \varphi(\mathbb{T}_x X)$. We consider for the rest of this section the degree 5 divisors

$$T := q^*\{x\} \subset C_L, \quad H := D_L \cap \varphi(\mathbb{T}_x X) = \overline{\varphi}_* T \subset D_L\,,$$

which can be assumed to be reduced by taking a generic $x \in L$. In other words, it holds

$$N_{\overline{\varphi}}\,\mathcal{O}_{C_L}(T) \cong N_{\overline{\varphi}}\,\mathcal{O}_{C_L}(\sigma T) \cong \mathcal{O}_{D_L}(H), \quad \overline{\varphi}^* \mathcal{O}_{D_L}(H) \cong \mathcal{O}_{C_L}(T + \sigma T), \quad T \cap \sigma T = \emptyset\,.$$

The plane quintic $D_L \subset \mathbb{P}(V/W)$ has canonical bundle $\omega_{D_L} \cong \mathcal{O}_{D_L}(2H)$, and so for

$$\kappa' := \mathcal{O}_{D_L}(H), \quad \kappa := \mathcal{O}_{C_L}(T + \sigma T)$$

it holds $\omega_{D_L} \cong \kappa'^2, \omega_{C_L} \cong \kappa^2$. We will verify below that $\mathrm{h}^0(\kappa)$ is even, so the theta characteristics κ and κ' are as in Lemma 3.9.

Lemma 4.1 *Every line bundle $\mathcal{L} \in \operatorname{Pic} D_L$ with $\deg \mathcal{L} \leq 5$ and $\mathrm{h}^0(\mathcal{L}) \geq 2$ is isomorphic to $\mathcal{O}_{D_L}(H - a + b)$ or $\mathcal{O}_{D_L}(H - a)$ for some $a \in H$, $b \in D_L$. In particular, D_L is not trigonal.*

Proof Write $\mathcal{L} \cong \mathcal{O}_{D_L}(E)$ for an effective divisor E. By Riemann-Roch $\mathrm{h}^0(\mathcal{L}^{-1}(2H)) = \mathrm{h}^0(\mathcal{L}) + 5 - \deg \mathcal{L}$, and thus $\dim |\mathcal{O}_{D_L}(2H - E)| \geq 6 - \deg E$. We view the linear system $|\mathcal{O}_{D_L}(2H - E)|$ as the set of conics in $\mathbb{P}(V/W)$ passing through the points E.

If $\deg E = 5$, there is a $\mathbb{P}^1$ of conics passing through E. But through any 5 points, no 4 of which are collinear, there passes a unique (possibly singular) conic. So 4 points of E must lie on a line, and E is therefore rationally equivalent to an $H - a + b$. If $\deg E = 4$, there is a $\mathbb{P}^2$ of conics passing through E. But the (possibly singular) conics through any 4 non-collinear points form a $\mathbb{P}^1$. So E must lie on a line, and is therefore rationally equivalent to an $H - a$. Similarly, the conics through any $k \leq 3$ points form a $\mathbb{P}^{5-k}$, so the cases $\deg E \leq 3$ cannot occur. □

Lemma 4.2 *It holds* $\mathrm{h}^0(\mathcal{O}_{C_L}(T)) = \mathrm{h}^0(\mathcal{O}_{C_L}(\sigma T)) = 2$, $\mathrm{h}^0(\mathcal{O}_{C_L}(T + \sigma T)) = 4$, *and the multiplication map is an isomorphism:*

$$\mathrm{H}^0(\mathcal{O}_{C_L}(T)) \otimes \mathrm{H}^0(\mathcal{O}_{C_L}(\sigma T)) \xrightarrow{\sim} \mathrm{H}^0(\mathcal{O}_{C_L}(T + \sigma T))\,.$$

Proof Let $\eta \in \operatorname{Pic}^0 D_L$ be the 2-torsion element induced by the double cover, so that

$$\mathrm{h}^0(\mathcal{O}_{C_L}(T + \sigma T)) = \mathrm{h}^0(f_* f^* \mathcal{O}_{D_L}(H)) = \mathrm{h}^0(\mathcal{O}_{D_L}(H)) + \mathrm{h}^0(\mathcal{O}_{D_L}(H) \otimes \eta)\,.$$

A direct computation shows $\mathrm{h}^0(\mathcal{O}_{D_L}(H)) = 3$, so for the dimensions it will be enough to verify the following two facts. First: $\mathrm{h}^0(\mathcal{O}_{D_L}(H) \otimes \eta) \leq 1$, because otherwise $\mathcal{O}_{D_L}(H) \otimes \eta \cong \mathcal{O}_{D_L}(H - a + b)$ by Lemma 4.1, and thus $\mathcal{O}_{D_L}(2a) \cong \mathcal{O}_{D_L}(2b) \otimes \eta^2 \cong \mathcal{O}_{D_L}(2b)$ would imply that D_L is hyperelliptic, which it is not. Second: $\mathrm{h}^0(\mathcal{O}_{C_L}(T + \sigma T)) \geq 2\,\mathrm{h}^0(\mathcal{O}_{C_L}(T)) \geq 4$. To see this, note that $\mathrm{h}^0(\mathcal{O}_{D_L}(T)) \geq \mathrm{h}^0(\mathcal{O}_L(x)) = 2$, so we can choose global sections $s, t \in \mathrm{H}^0(\mathcal{O}_{C_L}(\sigma T))$ without common zeros, which induce an exact sequence of sheaves

$$\mathcal{O}_{C_L}(T - \sigma T) \xhookrightarrow{(s\,\cdot\,, -t\,\cdot\,)} \mathcal{O}_{C_L}(T) \oplus \mathcal{O}_{C_L}(T) \xrightarrow{t\,\cdot\,+s\,\cdot} \mathcal{O}_{C_L}(T + \sigma T)\,.$$

Since $\deg T$ is odd, $\mathcal{O}_{C_L}(T)$ is not a pullback of a line bundle on D_L, which means that $\mathcal{O}_{C_L}(T) \not\cong \mathcal{O}_{C_L}(\sigma T)$, and thus $\mathrm{h}^0(\mathcal{O}_{C_L}(T - \sigma T)) = 0$. Therefore, the long exact sequence in cohomology implies $\mathrm{h}^0(\mathcal{O}_{C_L}(T + \sigma T)) \geq 2\,\mathrm{h}^0(\mathcal{O}_{C_L}(T)) \geq 4$.

The argument above shows that, for all $s \neq 0$, the multiplication map is injective:

$$\mathrm{H}^0(\mathcal{O}_{C_L}(T)) \xhookrightarrow{s\,\cdot} \mathrm{H}^0(\mathcal{O}_{C_L}(T + \sigma T)).$$

The same holds for σT, so the composed map on the tensor product is also injective. □

Lemma 4.3 *Every line bundle $\mathcal{L} \in \operatorname{Pic} C_L$ with $\mathrm{h}^0(\mathcal{L}) \geq 2$ and $N_{\overline{\varphi}}\mathcal{L} \cong \mathcal{O}_{D_L}(H)$ is isomorphic to $\mathcal{O}_{C_L}(T)$ or $\mathcal{O}_{C_L}(\sigma T)$.*

Proof It holds $\mathcal{O}_{C_L}(T + \sigma T) \cong \overline{\varphi}^* N_{\overline{\varphi}}\mathcal{L} \cong \mathcal{L} \otimes \sigma^*\mathcal{L}$. Analogously to the proof of Lemma 4.2, one shows that the multiplication map $\mathrm{H}^0(\mathcal{L}) \otimes \mathrm{H}^0(\sigma^*\mathcal{L}) \to \mathrm{H}^0(\mathcal{L} \otimes \sigma^*\mathcal{L})$ is injective, and thus an isomorphism by dimension reasons. Then, the induced morphism to projective space ϕ factors through the Segre embeddings induced by the multiplication maps

$$\begin{tikzcd}
 & \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{C_L}(T))^*) \times \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{C_L}(\sigma T))^*) \arrow[dr, hook, "i"] & \\
C_L \arrow[ur, "\phi_1"] \arrow[rr, "\phi"] \arrow[dr, "\phi_2"'] & & \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{C_L}(T + \sigma T))^*)\,. \\
 & \mathbb{P}(\mathrm{H}^0(\mathcal{L})^*) \times \mathbb{P}(\mathrm{H}^0(\sigma^*\mathcal{L})^*) \arrow[ur, hook, "j"'] &
\end{tikzcd}$$ □

By definition of the induced morphisms, we have

$$\mathcal{L} \cong \phi_2^*\mathcal{O}(1,0), \quad \mathcal{O}_{C_L}(T) \cong \phi_1^*\mathcal{O}(1,0), \quad \mathcal{O}_{C_L}(\sigma T) \cong \phi_1^*\mathcal{O}(0,1)\,,$$

where $\mathcal{O}(a, b)$ are the line bundles of type (a, b) on the respective product spaces. The images of i and j are quadrics which contain the curve $\phi(C_L)$. But quadrics in $\mathbb{P}^3$ form a

$\mathbb{P}^9$, so there is at most one containing the degree 10 curve $\phi(C_L)$. Thus, i factors through j to an isomorphism $\widetilde{i}$. Since any isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1 \times \mathbb{P}^1$ pulls back a bundle of type $(1,0)$ to one of type $(1,0)$ or $(0,1)$, it holds

$$\mathcal{L} \cong \phi_2^* \mathcal{O}(1,0) \cong \phi_1^* \widetilde{i}^* \mathcal{O}(1,0) \cong \phi_1^* \mathcal{O}(1,0) \text{ or } \phi_1^* \mathcal{O}(0,1) \cong \mathcal{O}_{C_L}(T) \text{ or } \mathcal{O}_{C_L}(\sigma T). \qquad \square$$

Lemma 4.4 *Every line bundle $\mathcal{L} \in \operatorname{Pic} C_L$ with $\mathrm{h}^0(\mathcal{L}) \geq 4$ and $N_{\overline{\varphi}}\mathcal{L} \cong \omega_{D_L}$ is isomorphic to $\mathcal{O}_{C_L}(T + \sigma T)$, $\mathcal{O}_{C_L}(2T)$ or $\mathcal{O}_{C_L}(2\sigma T)$.*

Proof Write $\mathcal{L} \cong \mathcal{O}_{C_L}(E)$ for an effective divisor E. Take global sections $s, t \in \mathrm{H}^0(\mathcal{O}_{C_L}(T))$ without common zeros, so that we have an exact sequence

$$\mathcal{O}_{C_L}(E - T) \xhookrightarrow{(s\,\cdot,\,-t\,\cdot)} \mathcal{O}_{C_L}(E) \oplus \mathcal{O}_{C_L}(E) \xrightarrow{t\,\cdot\,+s\,\cdot} \mathcal{O}_{C_L}(E + T),$$

and thus $\mathrm{h}^0(\mathcal{O}_{C_L}(E-T)) + \mathrm{h}^0(\mathcal{O}_{C_L}(E+T)) \geq 2\,\mathrm{h}^0(\mathcal{O}_{C_L}(E)) \geq 8$. By Riemann-Roch

$$\mathrm{h}^0(\mathcal{O}_{C_L}(E+T)) = \mathrm{h}^0(\omega_{C_L} \otimes \mathcal{O}_{C_L}(-E-T)) + 5 = \mathrm{h}^0(\mathcal{O}_{C_L}(\sigma E - T)) + 5,$$

where we have used that $\mathcal{O}_{C_L}(E + \sigma E) \cong \overline{\varphi}^* N_{\overline{\varphi}} \mathcal{O}_{C_L}(E) \cong \omega_{C_L}$. Combining, we obtain

$$\mathrm{h}^0(\mathcal{O}_{C_L}(E-T)) + \mathrm{h}^0(\mathcal{O}_{C_L}(E - \sigma T)) \geq 3.$$

Since $N_{\overline{\varphi}}\mathcal{O}_{C_L}(E-T) \cong N_{\overline{\varphi}}\mathcal{O}_{C_L}(E - \sigma T) \cong \mathcal{O}_{D_L}(H)$, it follows from Lemma 4.3 that

$$\mathcal{O}_{C_L}(E-T) \text{ or } \mathcal{O}_{C_L}(E-\sigma T) \text{ is isomorphic to } \mathcal{O}_{C_L}(T) \text{ or } \mathcal{O}_{C_L}(\sigma T).$$

So $\mathcal{O}_{C_L}(E)$ is isomorphic to $\mathcal{O}_{C_L}(T + \sigma T)$, $\mathcal{O}_{C_L}(2T)$ or $\mathcal{O}_{C_L}(2\sigma T)$. $\square$

4.2 Theta divisor of the cubic threefold

We showed in Lemma 2.31 that $\iota_L : C_L \hookrightarrow \mathrm{F}X$ induces an embedding of $\mathrm{H}^1(\mathrm{F}X, \mathbb{Z})$ into the Poincaré dual subgroup of $\ker \overline{\varphi}_*$, which is $\mathrm{H}^1(C_L, \mathbb{Z})^-$. This implied that the Hodge-Riemann pairing on FX was divisible by 6, thus delivering an isomorphism $\widehat{\mathrm{J}^3 X} \xrightarrow{p_* q^*} \widehat{\mathrm{J}^1 \mathrm{F}X}$ of Abelian varieties.

Theorem 4.5 (Mumford) *There is an isomorphism of Abelian varieties*

$$\left(\widehat{\mathrm{J}^3 X}, \mathcal{H}_X\right) \xrightarrow[\sim]{\iota_L^* p_* q^*} \left(\mathrm{P}(\widehat{C_L / D_L}), -\frac{1}{2}\mathcal{H}_{C_L}\right).$$

Proof Recall from Section 2.4 that the injection $\mathrm{H}^3(X,\mathbb{Z}) \xhookrightarrow{\iota_L^* p_* q^*} \mathrm{H}^1(C_L,\mathbb{Z})^-$ pulls back the alternating form $-\frac{1}{2}\Im\mathcal{H}_{C_L}$ to $\Im\mathcal{H}_X$, so $-\frac{1}{2}\Im\mathcal{H}_{C_L}$ is unimodular on the image of $\iota_L^* p_* q^*$. Since $\operatorname{rank}\mathrm{H}^3(X,\mathbb{Z}) = 10 = \operatorname{rank}\mathrm{H}^1(C_L,\mathbb{Z})^-$, the composition $\iota_L^* p_* q^*$ can only be an isomorphism. $\square$

Consider the theta characteristic $\kappa = \mathcal{O}_{C_L}(T+\sigma T)$ of Section 4.1, and the theta divisors $\Theta := \alpha_\kappa(W_{2g-2}(C_L))$ for $\widehat{\mathrm{J}\,C_L}$ and $\Xi := (\iota^*\Theta)_{\mathrm{red}}$ for $\mathrm{P}(\widehat{C_L/D_L})$ as per Lemma 3.11.

Theorem 4.6 $\mathcal{O}_{C_L} \in \Xi$ *is the unique singular point of* Ξ.

Proof By Corollary 3.12 there are two types of singularities. If $\mathcal{L} \in \Xi \subset \ker N_{\overline{\varphi}}$ has $\mathrm{h}^0(\mathcal{L}\otimes\kappa) \geq 4$, then Lemma 4.4 applies to $\mathcal{L}\otimes\kappa = \mathcal{L}(T+\sigma T)$, implying that $\mathcal{L}$ is isomorphic to $\mathcal{O}_{C_L}$, $\mathcal{O}_{C_L}(T-\sigma T)$ or $\mathcal{O}_{C_L}(\sigma T - T)$. However, since $\deg T \equiv 1 \bmod 2$, only the first one is contained in $\mathrm{P}(\widehat{C_L/D_L})$ by Proposition 3.8. On the other hand, by Proposition 3.13, exceptional singularities of Ξ are of the form $f^*\mathcal{M}\otimes\mathcal{N}\otimes\kappa^{-1}$, with $\mathrm{h}^0(\mathcal{M}) = 2$ and $\mathcal{M}^2 \otimes N_{\overline{\varphi}}\mathcal{N} \cong \omega_{D_L}$. Then $\deg\mathcal{M} = \frac{1}{2}(5-\deg\mathcal{N}) < 3$, which contradicts Lemma 4.1, so there are no singularities of this type. $\square$

Corollary 4.7 (Clemens-Griffiths) *X is not birational to $\mathbb{P}^3$.*

Proof If X were birational to $\mathbb{P}^3$, its intermediate Jacobian would have to be decomposable as an Abelian variety or isomorphic to the Jacobian of a curve, by Corollary 1.10. But a theta divisor Ξ for $\mathrm{J}^3 X$ has a unique singular point, so Ξ is irreducible and $\dim\operatorname{sing}\Xi < \dim \mathrm{J}^3 X - 4$, implying that $\mathrm{J}^3 X$ is indecomposable and not a Jacobian of a curve. $\square$

4.3 Albanese of the Fano surface

The theta divisor Ξ on $\mathrm{P}(\widehat{C_L/D_L})$ has been constructed in terms of the chosen line L. However, the polarizations on $\widehat{\mathrm{J}^3 X}$ and $\widehat{\mathrm{J}^1 \mathrm{F}X}$ are canonical; the latter is induced by the Plücker embedding $\mathrm{F}X \hookrightarrow \mathbb{P}(\bigwedge^2 V)$. In this section, we will give a canonical description of the theta divisors in terms of the Abel-Jacobi map $\Phi^0_{\mathrm{F}X,L_0}$ into the Albanese variety of $\mathrm{F}X$, which will be indipendent of the starting point $L_0 \in \mathrm{F}X$.

$$\begin{tikzcd}
\widehat{\mathrm{J}^3 X} \arrow[r, "p_*q^*", "\sim"'] & \widehat{\mathrm{J}^1 \mathrm{F}X} \arrow[r, "\iota_L^*", "\sim"'] \arrow[d, "\phi_{-\frac{1}{6}\mathcal{H}^1_{\mathrm{F}X}}", "\wr"'] & \mathrm{P}(\widehat{C_L/D_L}) \\
& \mathrm{J}^1 \mathrm{F}X &
\end{tikzcd}$$

Recall that $D_L \subset \mathbb{P}(V/W)$ is a quintic planar curve obtained by projecting from the line $L = \mathbb{P}W \subset \mathbb{P}V$. It maps every $\mathbb{P}^3 \subset \mathbb{P}V$ with $L \subset \mathbb{P}^3$ to a line in $\mathbb{P}(V/W)$, which can be seen as a divisor on D_L in the linear system $|H|$ of a line. We will consider the following morphism:

$$\mathrm{F}X \setminus \{L \cup C_L\} \xrightarrow{h\,:\,L' \mapsto \langle L, L'\rangle / W} \mathrm{Gr}_2(V/W) \xrightarrow[\sim]{\bullet \cap D_L} |H|\,. \tag{4.1}$$

Proposition 4.8 *Let $L' \in \mathrm{F}X \setminus (L \cup C_L)$ such that the cubic surface $S_{L'} := X \cap \langle L, L'\rangle$ is smooth and $h(L') \cap D_L$ is reduced. Then, there are precisely 5 lines $L_i \subset S_{L'}$ that intersect both L and L', and L, L' are the only 2 lines in $S_{L'}$ that intersect all 5 lines L_i. Moreover, the rational equivalence class of the resulting algebraic cycle $[2L + 2L' - \sum_{i=1}^{5} L_i] \in \mathrm{CH}_1(X)_{\mathrm{alg}}$ is indipendent of L'.*

Proof The divisor $h(L') \cap D_L$ consists of 5 distinct points a_i. Since L is of first type, the 5 reducible conics $\varphi^{-1}(a_i)$ contain 2 distinct lines (L_i, L_i') each, and every line in $X \cap \langle L, L'\rangle$ intersecting L is contained in one of these conics. Since $S_{L'}$ is assumed to be smooth, at most two lines in it intersect at any given point. Thus, without loss of generality, L' intersects L_i and not L_i' in each conic $\varphi^{-1}(a_i)$. Moreover, the 5 lines L_i intersect L in distinct points, so they are disjoint.

It is then a classical result of the theory of cubic surfaces (see for example [Huy23, **IV**.2.1]) that $S_{L'}$ can be blown down to $\mathbb{P}^2$ in such a way that the lines L_i, $i \leq 4$, are the first four out of six exceptional divisors E_i over points $x_i \in \mathbb{P}^2$. The intersection numbers are then completely described in terms of this minimal model, and they show that L and L' can only be the strict transforms of conics in $\mathbb{P}^2$ through $(x_i)_{i \neq 5}$ and $(x_i)_{i \neq 6}$, because they are the only ones that intersect the 4 exceptional divisors E_i, $i \leq 4$. Moreover, using the notation of [Huy23, **IV**.2.5], the 5-th line must be of type L_{56}, the strict transform of a line through x_5 and x_6. Then, if $C \subset S_{L'}$ is the strict transform of a line in the minimal model $\mathbb{P}^2$, it holds

$$L \sim 2C - \sum_{i \neq 5} E_i\,, \quad L' \sim 2C - \sum_{i \neq 6} E_i\,, \quad L_{56} \sim C - E_5 - E_6\,.$$

So we compute

$$2L + 2L' + \sum_{i=1}^{4} E_i + L_{56} \sim 9C - 3\sum_{i=1}^{6} E_i \sim 3\, S_{L'} \cap P\,,$$

where $S_{L'} \cap P \sim 3C - \sum_{i=1}^{6} E_i$ is a hyperplane section of $S_{L'}$, whose rational equivalence class in $\mathrm{CH}_1(X)$ is indipendent of L'. □

Corollary 4.9 *Let $L_0 \in \mathrm{FX} \setminus (L \cup C_L)$ with S_{L_0} smooth and $h(L_0) \cap D_L$ reduced. Then, associating to each L' (with $S_{L'}$ smooth and $h(L') \cap D_L$ reduced) the effective divisor $C_L \cap C_{L'}$ defines a rational map that fits with the Abel-Jacobi map Φ^0_{FX,L_0} in the commutative diagram*

$$\begin{tikzcd}
\mathrm{FX} \arrow[r, dashed, "L' \mapsto C_L \cap C_{L'}"] \arrow[d, "\Phi^0_{\mathrm{FX},L_0}"] & C_L^{(5)} \arrow[d, "-\Phi^0_{C_L, C_L \cap C_{L_0}}"] \\
\mathrm{J}^1\,\mathrm{FX} & \mathrm{J}\,C_L \\
\widehat{\mathrm{J}^1\,\mathrm{FX}} \arrow[u, "\phi_{-\frac{1}{6}\mathcal{H}^1_{\mathrm{FX}}}", "\wr"'] \arrow[r, hook, "\iota_L^*"] & \widehat{\mathrm{J}\,C_L} \arrow[u, "\phi_{-\mathcal{H}_{C_L}}", "\wr"']
\end{tikzcd}$$

Proof Interpreting $C_L \cap C_{L'}$ simultaneously as an element of $\mathrm{CH}_1(X)$ and of $\mathrm{CH}_0(C_L)$, it holds $C_L \cap C_{L'} = \sum_i L_i$ in the notation of Lemma 4.8 and

$$2[L'] - 2[L_0] = -[C_L \cap C_{L'}] + [C_L \cap C_{L_0}] = q_* p^* \iota_{L*}(-[C_L \cap C_{L'}] + [C_L \cap C_{L_0}]) \in \mathrm{CH}_1(X)_{\mathrm{alg}}.$$

Since $q_* p^* : \mathrm{J}^1\,\mathrm{FX} \to \mathrm{J}^3\,X$ is an isomorphism, being the dual of $p_* q^*$, it follows

$$2\Phi^0_{\mathrm{FX},L_0}(L') = 2\Phi^0_{\mathrm{FX}}([L' - L_0]) = -\iota_{L*}\Phi^0_{C_L}([C_L \cap C_{L'} - C_L \cap C_{L_0}]) = -\iota_{L*}\Phi^0_{C_L, C_L \cap C_{L_0}}(C_L \cap C_{L'}).$$

Identifying through the dualizing isomorphisms, and recalling from Lemma 2.31 that the Hermitian form $\mathcal{H}_{C_L}$ is pulled back to $2 \cdot \frac{1}{6}\mathcal{H}^1_{\mathrm{FX}}$ through ι_L^*, it holds

$$2\iota_L^* \Phi^0_{\mathrm{FX},L_0}(L') = -\iota_L^* \iota_{L*} \Phi^0_{C_L, C_L \cap C_{L_0}}(C_L \cap C_{L'}) = -2\Phi^0_{C_L, C_L \cap C_{L_0}}(C_L \cap C_{L'})\,.$$

Both sides are continuous maps from a connected domain, and the kernel of multiplication by 2 on $\widehat{\mathrm{J}\,C_L}$ is a discrete subset, so their difference must be constant. Then, since the two maps agree on L_0, they must agree everywhere. $\square$

Theorem 4.10 *The Abel-Jacobi map Φ^0_{FX,L_0} with respect to any $L_0 \in \mathrm{FX}$ is an embedding.*

Proof We can assume $L_0 \notin \{L \cup C_L\}$ with S_{L_0} smooth and $h(L_0) \cap D_L$ reduced. By Corollary 4.9, the image of the rational map $L' \mapsto C_L \cap C_{L'}$ lies in the preimage

$$P := \left\{ D + C_L \cap C_{L_0} \in C_L^{(5)} \;\middle|\; \substack{D \in \mathrm{Div}^0 C_L, \\ \overline{\varphi}_* D \sim_{\mathrm{rat}} 0} \right\} \subset C_L^{(5)}$$

of $\ker N_\varphi$ under the Abel-Jacobi map $D \mapsto \mathcal{O}_{C_L}(D - C_L \cap C_{L_0})$. By Proposition 3.8, P decomposes in two disjoint subspaces P_0 and P_1 mapping to $\mathrm{P}(\widehat{C_L/D_L}) \subset \ker N_{\overline{\varphi}}$ and its unique coset respectively.

$$\begin{tikzcd}
\mathrm{F}X \arrow[r, dashed, "L' \mapsto C_L \cap C_{L'}"] \arrow[d, "\Phi^0_{\mathrm{F}X, L_0}"] & P_0 \arrow[dd] & \subset & P \arrow[dd] & \subset & C_L^{(5)} \arrow[d, "-\Phi^0_{C_L, C_L \cap C_{L_0}}"] \\
\mathrm{J}^1\,\mathrm{F}X & & & & & \mathrm{J}\,C_L \\
\widehat{\mathrm{J}^1\,\mathrm{F}X} \arrow[u, "\phi_{-\frac{1}{6}\mathcal{H}^1_{\mathrm{F}X}}", "\wr"'] \arrow[r, "\iota_L^*", "\sim"'] & \mathrm{P}(\widehat{C_L/D_L}) & \subset & \ker N_{\overline{\varphi}} & \subset & \widehat{\mathrm{J}\,C_L} \arrow[u, "\phi_{-\mathcal{H}_{C_L}}", "\wr"']
\end{tikzcd} \tag{4.2}$$

To verify that $\Phi^0_{\mathrm{F}X, L_0}$ is injective, we want to relate the rational map $L' \mapsto C_L \cap C_{L'}$ to the morphism $P_0 \to \mathrm{P}(\widehat{C_L/D_L})$.

The projection h from diagram (4.1) can be extended to $\mathrm{F}X \setminus L$ by sending a line L' with $L \cap L' = \{x\}$ to the projection $\mathbb{T}_x X/W$ of the embedded tangent space. For every $L' \in \mathrm{F}X \setminus \{L \cup C_L\}$, the pushforward $\overline{\varphi}_*(C_L \cap C_{L'}) = h(L') \cap D_L$ is a divisor on D_L in the linear system $|H|$ of a line $H \subset \mathbb{P}(V/W)$, so this holds also on $\mathrm{F}X \setminus L$ by continuity. By blowing up at L it also extends to a morphism from $\mathrm{Bl}_L\,\mathrm{F}X$. Denote $\Psi : \mathrm{Bl}_L\,\mathrm{F}X \dashrightarrow P_0$ the rational map $L' \mapsto C_L \cap C_{L'}$ lifted to the blow-up, which fits with h and $\overline{\varphi}_*$ in diagram (4.3). We claim that Ψ actually extends to an isomorphism. To this end, we notice two facts:

- $\overline{\varphi}_*$ is finite of degree 16, because $\overline{\varphi}$ is a double cover and thus any divisor $\sum_{i=1}^5 a_i \in |H|$ has at most $\frac{1}{2}2^5$ preimages (the bound being achieved for distinct a_i).

- h is finite of degree 16. Indeed, given any $L \subset \mathbb{P}^3 \subset \mathbb{P}V$, the cubic surface $X \cap \mathbb{P}^3$ contains only finitely many lines because X does not contain any plane (see Lemma 1.16), so the fibers in $\mathrm{F}X \setminus L$ are finite. Since L is a line of first type, the exceptional divisor $\mathbb{P}(\mathrm{T}_L\,\mathrm{F}X) \subset \mathrm{Bl}_L\,\mathrm{F}X$ also adds only finitely many elements to each fiber. Moreover, each fiber over the dense open set $\mathrm{Gr}_2(V/W) \setminus (h(\mathbb{P}(\mathrm{T}_L\,\mathrm{F}X)) \cup h(C_L))$ consists of the lines in the cubic surface different from L and not intersecting L, of which there are at most $27 - 10 - 1 = 16$.

In particular, $\overline{\varphi}_*$ and h are proper. Thus, since $\mathrm{Bl}_L\,\mathrm{F}X$ is normal, the valuative criterions for properness imply that Ψ extends to a morphism, which is a finite morphism because $h = \overline{\varphi}_* \circ \Psi$ is. By Lemma 4.8, for any two lines $L', L'' \in \mathrm{F}X \setminus (L \cup C_L)$ such that $S_{L'} = S_{L''}$ are smooth, if the 5 lines in $S_{L'}$ intersecting both L and L' are equal to the 5 lines in $S_{L''}$ intersecting both L and L'', then $L' = L''$. In other words, Ψ is injective over the open dense subset of lines whose induced cubic surface is smooth.

It follows that Ψ is the normalisation of an irreducible component of P_0. We refer to [Bea82, §2] for the proof that P_0 is normal and irreducible, so that Ψ is an isomorphism.

$$\begin{tikzcd} \mathrm{Bl}_L \mathrm{FX} \arrow[r, two heads, "h"] \arrow[d, two heads] \arrow[drr, "\Psi"', "\sim"] & \mathrm{Gr}_2(V/W) \arrow[r, "\bullet\cap D_L", "\sim"'] & {|H|} \\ \mathrm{FX} \arrow[rr, dashed, "L' \mapsto C_L \cap C_{L'}"] & & P_0 \subset P \subset C_L^{(5)} \arrow[u, two heads, "\overline{\varphi}_*"'] \end{tikzcd} \tag{4.3}$$

Now we look at diagrams (4.2) and (4.3) together. Lemma 4.3 shows that the only fibers of $P \to \ker N_{\overline{\varphi}}$ containing more than one divisor are the two 1-dimensional linear systems of the bundles $q^*\mathcal{O}_L(1)$ and $\sigma^* q^* \mathcal{O}_L(1)$, one in P_0 and its involute in P_1. So the composed map $\mathrm{Bl}_L \mathrm{FX} \to \mathrm{P}(\widehat{C_L/D_L})$ blows down a line and is injective outside of that line. Then, this line must be the exceptional divisor of $\mathrm{Bl}_L \mathrm{FX}$, meaning that

$$\iota_L^*(\,\Phi^0_{\mathrm{FX},L_0}(L)\,) \;\cong\; q^*\mathcal{O}_L(-1) \otimes \mathcal{O}_{C_L}(C_L \cap C_{L_0}) \;\in \mathrm{P}(\widehat{C_L/D_L}),$$

and thus Φ^0_{FX,L_0} is injective everywhere. Moreover, since $\Phi^0_{C_L, C_L \cap C_{L_0}}$ is birational onto its image and Φ^0_{FX,L_0} is the blow-down of it, Φ^0_{FX,L_0} is an isomorphism onto its image. $\square$

Theorem 4.11 (Clemens-Griffiths, Tyurin) *There is an isomorphism between the tangent bundle and the tautological bundle*

$$\mathcal{T}_{\mathrm{FX}} \xrightarrow{\sim} \mathcal{S}_{\mathrm{FX}}\,.$$

Proof Our approach makes use of the classical theory laid out in Section 2.5, which follows closely [Tyu70]. For a more algebraic proof, see [AK77], [Huy23, **V**.2.2]. Since the cotangent bundle $\mathcal{T}^*_{\mathrm{J}^1\mathrm{FX}}$ is trivial and the differential $\mathrm{d}\,\Phi^0_{\mathrm{FX},L_0} : \mathcal{T}_{\mathrm{FX}} \to \mathcal{T}_{\mathrm{J}^1\mathrm{FX}}$ is injective at every fiber over $L' \in \mathrm{FX}$, each restriction of the dual is surjective:

$$\begin{tikzcd} \mathrm{H}^0(\mathrm{J}^1\mathrm{FX}, \mathcal{T}^*_{\mathrm{J}^1\mathrm{FX}}) \arrow[r, "(\Phi^0_{\mathrm{FX},L_0})^*", "\sim"'] \arrow[d, "\wr"] & \mathrm{H}^0(\mathrm{FX}, \mathcal{T}^*_{\mathrm{FX}}) \arrow[d, two heads, "\text{restriction}"] \\ \mathrm{T}^*_{\Phi^0_{\mathrm{FX},L_0}(L')}(\mathrm{J}^1\mathrm{FX}) \arrow[r, two heads, "\mathrm{d}^*_{L'}\,\Phi^0_{\mathrm{FX},L_0}"] & \mathrm{T}^*_{L'}\mathrm{FX} \end{tikzcd}$$

Since $\omega_{\mathrm{FX}} \cong \det \mathcal{S}^*_{\mathrm{FX}}$ by Lemma 2.28, the isomorphism is delivered by Theorem 2.27. $\square$

Corollary 4.12 *Consider the trivialization of the tangent bundle*

$$\mathcal{T}_{\mathrm{J}^1\mathrm{FX}} \xrightarrow{\sim} \mathrm{J}^1\mathrm{FX} \times \mathrm{H}^{1,0}(\mathrm{FX})^* \twoheadrightarrow \mathrm{H}^{1,0}(\mathrm{FX})^*\,.$$

The isomorphism of Theorem 4.11 fits with the projectivized differential of the Abel-Jacobi map $\mathbb{P}(\mathrm{d}\,\Phi^0_{\mathrm{FX},L_0})$ *and the projection q in the following commutative diagram:*

$$\begin{tikzcd}
 & \mathbb{P}\mathcal{T}_{\mathrm{J}^1\,\mathrm{FX}} \arrow[dr, two heads, bend left] & \\
\mathbb{P}\mathcal{T}_{\mathrm{FX}} \arrow[ur, hook, bend left, "\mathbb{P}(\mathrm{d}\,\Phi^0_{\mathrm{FX},L_0})"] \arrow[r, "\varphi_{\mathcal{S}^*_{\mathbb{P}\mathcal{T}_{\mathrm{FX}}}}"] \arrow[d, "\wr"'] & \mathbb{P}(\mathrm{H}^0(\mathbb{P}\mathcal{T}_{\mathrm{FX}}, \mathcal{S}^*_{\mathbb{P}\mathcal{T}_{\mathrm{FX}}})^*) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mathbb{P}(\mathrm{H}^0(\mathrm{FX}, \mathcal{T}^*_{\mathrm{FX}})^*) \arrow[d, "\wr"', "\text{Theorem 2.27}"] \\
\mathbb{L} = \mathbb{P}\mathcal{S}_{\mathrm{FX}} \arrow[r, "\varphi_{\mathcal{S}^*_{\mathbb{L}}}"] \arrow[d, two heads, "q"'] & \mathbb{P}(\mathrm{H}^0(\mathbb{L}, \mathcal{S}^*_{\mathbb{L}})^*) \arrow[r, "\sim"] \arrow[d, "\wr"', "(q^*)^*" ,"\text{Lemma 2.21}"'] & \mathbb{P}(\mathrm{H}^0(\mathrm{FX}, \mathcal{S}^*_{\mathrm{FX}})^*) \\
X \arrow[r, hook] & \mathbb{P}V &
\end{tikzcd}$$

For all $L' = \mathbb{P}W' \in \mathrm{FX}$, the diagram restricts to isomorphisms

$$\begin{tikzcd}
\mathcal{T}_{\mathrm{FX}}\,|_{L'} = \mathrm{T}_{L'}\,\mathrm{FX} \arrow[d, "\wr"] \arrow[dr, bend left, "\mathrm{d}_{L'}\,\Phi^0_{\mathrm{FX},L_0}", "\sim"'] & & \\
\mathcal{S}_{\mathrm{FX}}\,|_{L'} \arrow[r, "q", "\sim"'] & W' \arrow[r, hook] & V
\end{tikzcd}$$

Proof The only part that we have not verified before is the compatibility with the Abel-Jacobi map. By construction, for $v \in \mathrm{T}_{L'}\,\mathrm{FX}$ it holds

$$\mathrm{d}_{L'}\,\Phi^0_{\mathrm{FX},L_0}(v) = \Big(\alpha \in \mathrm{H}^{1,0}(\mathrm{FX}) \mapsto \alpha|_{L'}(v) \in \mathbb{C}\Big) \ \in \mathrm{H}^{1,0}(\mathrm{FX})^* = \mathrm{T}_{\Phi^0_{\mathrm{FX},L_0}(L')}(\mathrm{J}^1\,\mathrm{FX}).$$

Projectivizing, for $(L', [v]) \in \mathbb{P}\mathcal{T}_{\mathrm{FX}}$ we get

$$\begin{aligned}
\varphi_{\mathcal{S}^*_{\mathbb{P}\mathcal{T}_{\mathrm{FX}}}}(L', [v]) &= \Big[\alpha \ \mapsto \alpha|_{(L',[v])} \in \mathcal{S}^*_{\mathbb{P}\mathcal{T}_{\mathrm{FX}}}|_{(L',[v])}\Big] && \in \mathbb{P}(\mathrm{H}^0(\mathbb{P}\mathcal{T}_{\mathrm{FX}}, \mathcal{S}^*_{\mathbb{P}\mathcal{T}_{\mathrm{FX}}})^*) \\
&= [\alpha \ \mapsto \alpha|_{L'}(v) \in \mathbb{C}] = \big[\mathrm{d}_{L'}\,\Phi^0_{\mathrm{FX},L_0}(v)\big] && \in \mathbb{P}(\mathrm{H}^0(\mathrm{FX}, \mathcal{T}^*_{\mathrm{FX}})^*). \qquad \square
\end{aligned}$$

Lemma 4.13 *The image of the difference map*

$$\delta \ : \ \mathrm{FX} \times \mathrm{FX} \xrightarrow{(\Phi^0_{\mathrm{FX},L_0},\, \Phi^0_{\mathrm{FX},L_0})} \mathrm{J}^1\,\mathrm{FX} \times \mathrm{J}^1\,\mathrm{FX} \xrightarrow{\ -\ } \mathrel{\mkern-14mu}\rightarrow \mathrm{J}^1\,\mathrm{FX}\,,$$

which is indipendent of L_0, is the theta divisor Ξ.

Proof There is an analogous difference map for the morphism $P_0 \to \mathrm{P}(\widehat{C_L/D_L})$ from the proof of Theorem 4.10, which fits with our difference map δ in diagram (4.4). We claim that the image of $P_0 \times P_0 \to \widehat{\mathrm{J}\,C_L}$ is contained in the theta divisor. Indeed, a divisor

$D + C_L \cap C_{L_0} \in P_0$ is effective and satisfies $\overline{\varphi}_* D \sim_{\mathrm{rat}} 0$, $\mathcal{O}_{C_L}(-D) \cong \mathcal{O}_{C_L}(\sigma D)$. Since $\overline{\varphi}_*(C_L \cap C_{L_0}) \sim_{\mathrm{rat}} H$ is a hyperplane, the theta characteristic of Section 4.1 is

$$\kappa = \mathcal{O}_{C_L}(T + \sigma T) \cong \mathcal{O}_{C_L}(C_L \cap C_{L_0} + \sigma(C_L \cap C_{L_0})),$$

which makes

$$\mathcal{O}_{C_L}(\sigma D + D') \otimes \kappa \cong \mathcal{O}_{C_L}(\sigma(D + C_L \cap C_{L_0}) + (D' + C_L \cap C_{L_0}))$$

an effective line bundle.

$$\begin{tikzcd}
\mathrm{Bl}_L \mathrm{FX} \times \mathrm{Bl}_L \mathrm{FX} \arrow[r, "\Psi\times\Psi", "\sim"'] \arrow[d, two heads] & P_0 \times P_0 \arrow[ddd] \arrow[ddrr, "\substack{(D+C_L\cap C_{L_0},\, D'+C_L\cap C_{L_0}) \\ \mapsto \\ \mathcal{O}_{C_L}(-D+D') \cong \mathcal{O}_{C_L}(\sigma D+D')}"] & & \\
\mathrm{FX} \times \mathrm{FX} \arrow[d, "\delta"] & & & \\
\mathrm{J}^1 \mathrm{FX} & & \widehat{\mathrm{J}\, C_L} & \Theta \arrow[l, hook'] \\
\widehat{\mathrm{J}^1 \mathrm{FX}} \arrow[u, "\wr", "\Phi_{-\frac{1}{6}\mathcal{H}^1_{\mathrm{FX}}}"'] \arrow[r, "\iota_L^*", "\sim"'] & \mathrm{P}(\widehat{C_L/D_L}) \arrow[ur, hook] & \Xi \arrow[l, hook'] \arrow[ur, hook] &
\end{tikzcd} \tag{4.4}$$

By Corollary 4.12, the differential

$$\mathrm{d}_{(L',L'')}\,\delta \;:\; \mathrm{T}_{L'}\,\mathrm{FX} \oplus \mathrm{T}_{L''}\,\mathrm{FX} \xrightarrow{\mathrm{d}_{L'}\,\Phi^0_{\mathrm{FX},L_0} - \mathrm{d}_{L''}\,\Phi^0_{\mathrm{FX},L_0}} \mathrm{T}_{\Phi^0_{\mathrm{FX}}(L'-L'')}\,\mathrm{J}^1\,\mathrm{FX}$$

is of rank 4, 3 or 2, depending upon wether $L' \cap L'' = \emptyset$, $L' \cap L'' = \{*\}$ or $L' = L''$ respectively. So δ is generically finite and its image is 4-dimensional. Since it is contained in the irreducible divisor Ξ, it must be already equal to Ξ. $\square$

Theorem 4.14 *The projectivized tangent cone* $\mathbb{P}(\mathrm{TC}_0\,\Xi) \subset \mathbb{P}(\mathrm{T}_0\,\mathrm{J}^1\,\mathrm{FX})$ *is a cubic threefold isomorphic to X. In particular, the unique singularity* $0 \in \Xi$ *is a triple point.*

Proof The projectivized tangent cone at 0 can be described as the exceptional divisor of the blowup $\mathrm{Bl}_0\,\Xi$. Since Φ^0_{FX,L_0} is injective, the preimage of $0 \in \Xi$ under the difference map δ of Lemma 4.13 is the diagonal $\Delta \subset \mathrm{FX} \times \mathrm{FX}$. Then, the universal property of blowing-up [Har77, **II**.7.15] induces a commutative diagram

$$\begin{tikzcd}
& \mathrm{Bl}_\Delta(\mathrm{FX}\times\mathrm{FX}) \arrow[rr, two heads, "\widetilde{\delta}"] \arrow[dd, two heads] & & \mathrm{Bl}_0\,\Xi \arrow[rr, hook] \arrow[dd, two heads] & & \mathrm{Bl}_0(\mathrm{J}^1\,\mathrm{FX}) \arrow[dd, two heads] \\
\mathbb{P}(\mathcal{N}_{\Delta/\mathrm{FX}\times\mathrm{FX}}) \arrow[ur, hook] \arrow[rr, two heads, "\widetilde{\delta}"] \arrow[dd, two heads] & & \mathbb{P}(\mathrm{TC}_0\,\Xi) \arrow[ur, hook] \arrow[rr, hook] \arrow[dd, two heads] & & \mathbb{P}(\mathrm{T}_0\,\mathrm{J}^1\,\mathrm{FX}) \arrow[ur, hook] \arrow[dd, two heads] & \\
& \mathrm{FX}\times\mathrm{FX} \arrow[rr, two heads, "\delta"] & & \Xi \arrow[rr, hook] & & \mathrm{J}^1\,\mathrm{FX} \\
\Delta \arrow[ur, hook] \arrow[rr, two heads, "\delta"] & & \{0\} \arrow[ur, hook] \arrow[rr, equal] & & \{0\}\,. \arrow[ur, hook] &
\end{tikzcd} \tag{4.5}$$

The morphisms $\widetilde{\delta}$ are surjective because they have maximal image dimension in irreducible varieties. They send a normal direction $((L,L),[v,w])$ at a point $(L,L) \in \Delta$ to the projectivized vector $[\mathrm{d}\,\Phi^0_{\mathrm{FX},L_0}(v) - \mathrm{d}\,\Phi^0_{\mathrm{FX},L_0}(w)] \in \mathbb{P}(\mathrm{T}_0\,\mathrm{J}^1\,\mathrm{FX})$, so the following diagram commutes

$$\begin{tikzcd}
\mathbb{P}(\mathcal{N}_{\Delta/\mathrm{FX}\times\mathrm{FX}}) \arrow[r, two heads, "\widetilde{\delta}"] & \mathbb{P}(\mathrm{TC}_0\,\Xi) \arrow[r, hook] & \mathbb{P}(\mathrm{T}_0\,\mathrm{J}^1\,\mathrm{FX}) \\
\mathbb{P}\mathcal{T}_{\mathrm{FX}} \arrow[u, "\wr"', "\substack{((L,L),[\frac{v}{2},\frac{-v}{2}]) \\ \uparrow \\ (L,[v])}"] \arrow[r, "\mathbb{P}(\mathrm{d}\,\Phi^0_{\mathrm{FX},L_0})"] & \mathbb{P}\mathcal{T}_{\mathrm{J}^1\,\mathrm{FX}} \arrow[r, two heads] & \mathbb{P}(\mathrm{H}^0(\mathrm{FX},\mathcal{T}^*_{\mathrm{FX}})^*) \arrow[u, equal]
\end{tikzcd}$$

Combining with the diagram of Corollary 4.12, we see that the projectivized tangent cone is isomorphic to the image of $q : \mathbb{L} := \mathbb{P}\mathcal{S}_{\mathrm{FX}} \to \mathbb{P}V$, which is X. □

An immediate consequence is the Torelli theorem for cubic threefolds. However, it should be mentioned that the original proof in [CG72] follows the steps of Andreotti's proof of the Torelli theorem for curves, and not our description of the tangent cone.

Corollary 4.15 (Clemens-Griffiths) *Two smooth cubic threefolds $X, X' \subset \mathbb{P}^4$ are isomorphic if and only if their intermediate Jacobians $(\mathrm{J}^3\,X, \mathcal{H}_X)$ and $(\mathrm{J}^3\,X', \mathcal{H}_{X'})$ are isomorphic as Abelian varieties.*

Proposition 4.16 *The difference map $\delta : \mathrm{FX} \times \mathrm{FX} \twoheadrightarrow \Xi$ is generically finite of degree 6. For $L, L' \in \mathrm{FX} \setminus (\mathcal{D}_2 \cup \mathcal{R}_2)$ non-intersecting, with $X \cap \langle L, L' \rangle$ smooth and $C_L \cap C_{L'} = (L_i)_i$ consisting of 5 distinct lines, the fiber over $\delta(L,L')$ contains the 6 tuples*

$$\{\ (L,L'),\ (\,\sigma' L_i\,,\,\sigma L_i\,)_{i\leq 5}\ \} \overset{\delta}{\mapsto} \{\ \delta(L,L')\ \},$$

where σ, σ' are the involutions of $C_L, C_{L'}$ over $D_L, D_{L'}$ respectively. A generic fiber is of this type.

Proof Looking back at the blow-up diagram of Theorem 4.14, the blown-up map $\widetilde{\delta}$: $\mathbb{P}(\mathcal{N}_{\Delta/\mathrm{FX}\times\mathrm{FX}}) \twoheadrightarrow \mathbb{P}(\mathrm{TC}_0\,\Xi)$ is finite of degree 6 because it coincides with the projection q. Then, $\widetilde{\delta} : \mathrm{Bl}_\Delta(FX \times FX) \twoheadrightarrow \mathrm{Bl}_0\,\Xi$ must be generically finite of the same degree, and the same holds for δ.

By Lemma 4.8, there are indeed 5 unique lines L_i in X intersecting both L and L'. Each triple $(L, L_i, \sigma L_i)$, $(L', L_i, \sigma' L_i)$ is coplanar, so

$$[L + L_i + \sigma L_i] = [X \cap \mathbb{P}^2] = [L' + L_i + \sigma' L_i] \ \in \mathrm{CH}_1(X)_{\mathrm{alg}}, \quad \text{for all } i \leq 5,$$

for some $\mathbb{P}^2 \subset \mathbb{P}V$. After applying $p_* q^*$ on the level of Chow groups, it holds

$$\delta(L,L') = \Phi^0_{\mathrm{FX}}([L - L']) = \Phi^0_{\mathrm{FX}}([\sigma' L_i - \sigma L_i]) = \delta(\sigma' L_i, \sigma L_i). \qquad \square$$

Appendix A

Complex tori and Abelian varieties

Let V be a complex vector space of dimension g and $\Lambda \subset V$ a complete lattice, so that the quotient $X = V/\Lambda$ is a compact complex manifold of dimension g, called a *complex torus*.

A.1 Hodge decomposition and group cohomology

The complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ decomposes in two complex vector spaces $V^{1,0} \oplus V^{0,1}$ with $V \cong V^{1,0} \cong \overline{V^{0,1}}$ over $\mathbb{C}$. The groups of alternating forms on $V_{\mathbb{C}}$ decompose accordingly as

$$\begin{array}{ccccccc}
\mathrm{Alt}^k(\Lambda,\mathbb{Z}) & \xleftarrow{\sim} & \bigwedge^k \mathrm{Hom}(\Lambda,\mathbb{Z}) & & & & \\
\big\downarrow & & \big\downarrow & & & & \\
\mathrm{Alt}^k(\Lambda,\mathbb{C}) & \xleftarrow{\sim} & \bigwedge^k \mathrm{Hom}(\Lambda,\mathbb{C}) & & & & \\
\big\downarrow \wr & & \big\downarrow \wr & & & & \\
\mathrm{Alt}^k_{\mathbb{C}}(V_{\mathbb{C}},\mathbb{C}) & \xleftarrow{\sim} & \bigwedge^k \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}},\mathbb{C}) & \xleftarrow{\sim} & \bigwedge^k \Omega \oplus \overline{\Omega} & \xleftarrow{\sim} & \bigoplus_{p+q=k} \Omega^p \otimes \overline{\Omega}^q ,
\end{array}$$

where $\Omega = \mathrm{Hom}_{\mathbb{C}}(V^{1,0},\mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ and $\overline{\Omega} = \mathrm{Hom}_{\mathbb{C}}(V^{0,1},\mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(\overline{V},\mathbb{C})$. It is then the content of the Hodge theorem [BL04, 1.4] that the maps extending (p,q)-alternating forms to (p,q)-differential forms on X induce isomorphisms $\Omega^p \otimes \overline{\Omega}^q \xrightarrow{\sim} \mathrm{H}^q(X,\Omega^p_X)$ to (p,q)-Dolbeault cohomolgy, where Ω^p_X is the sheaf of holomorphic p-forms on X. Thus, we obtain the Hodge decomposition.

$$\begin{tikzcd}
\mathrm{Alt}^k_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) \arrow[d, "\wr"] & \bigoplus\limits_{p+q=k} \Omega^p \otimes \overline{\Omega}^q \arrow[l, "\sim"'] \arrow[d, "\wr"] \\
\mathrm{H}^k(X, \underline{\mathbb{C}}) & \bigoplus\limits_{p+q=k} \mathrm{H}^q(X, \Omega^p_X)
\end{tikzcd}$$

Computing automorphy factors of primitives of $(0,1)$-forms on V with respect to the deck transformation group of the universal cover gives an isomorphism to group cohomology

$$\begin{tikzcd}
\mathrm{H}^1(X, \underline{\mathbb{C}}) \arrow[r, "{[\omega] \mapsto [\lambda \mapsto \int_0^\lambda \pi^*\omega] = \left[\lambda \mapsto \underset{f \in C^\infty_V(V) \text{ with } df = \pi^*\omega}{f \circ \sigma_\lambda - f}\right]}", "\sim"'] \arrow[d, two heads, "\begin{array}{c}\text{sheaf}\\ \text{inclusion}\\ \shortparallel\\ \text{Hodge}\\ \text{projection}\end{array}"] & \mathrm{H}^1(\Lambda, \mathbb{C}) \arrow[d, two heads, "\text{inclusion}"] \\
\mathrm{H}^1(X, \mathcal{O}_X) \arrow[r, "{[\omega] \mapsto \left[\lambda \mapsto \underset{f \in C^\infty_V(V) \text{ with } \bar\partial f = \pi^*\omega}{f \circ \sigma_\lambda - f}\right]}", "\sim"'] & \mathrm{H}^1(\Lambda, \mathcal{O}_V(V))
\end{tikzcd}$$

where, for each $\lambda \in \Lambda \cong \pi_1(X)$, σ_λ is the induced deck transformation on $\pi : V \to X$.

A.2 Polarized complex tori

Definition A.1 *A non-degenerate Hermitian form H on V, whose imaginary part is integer-valued on Λ, is called a* polarization *of the complex torus V/Λ. The polarization is called* principal *if the alternating form $\Im H$ on Λ is unimodular. If in addition H is positive-definite, then the polarized complex torus $(V/\Lambda, H)$ is called an* Abelian variety. [1]

Positive-definite polarizations are special because of the following result [BL04, 4.5].

Theorem A.2 *A complex torus is projective if and only if it admits a positive-definite polarization, that is if it has the structure of an Abelian variety.*

Definition A.3 *A* morphism of polarized complex tori $(V/\Lambda, H) \to (V'/\Lambda', H')$ *is a linear homomorphism $f : V \to V'$ with $f(\Lambda) \subseteq \Lambda'$ and which pulls back H' to H by precomposition; so f is necessarily injective. If f is surjective and has finite kernel, the induced morphism is called an* isogeny.

[1] In our primary reference [BL04, 4.1], and generally in more modern terminology, a polarization is already assumed to be positive-definite. We use the definition given in [CG72, 3.5] because it will be convenient to consider the more general category of polarized complex tori, with polarizations of any definiteness.

Definition A.4 *The cartesian product of polarized complex tori* $(V/\Lambda, H) \times (V'/\Lambda', H')$ *is naturally a polarized complex torus* $((V \oplus V')/(\Lambda \oplus \Lambda'), H \oplus H')$. *Any subtorus* $V/\Lambda \subseteq V'/\Lambda'$ *of a polarized complex torus* $(V'/\Lambda', H')$ *has the natural restricted polarization* $H'|_V$, *making* $(V/\Lambda, H'|_V) \hookrightarrow (V'/\Lambda', H')$ *an embedding of polarized complex tori.*

Definition A.5 *The* dual complex torus *of* V/Λ *can be defined as* [2]

$$\widehat{V/\Lambda} := \widehat{V}/\widehat{\Lambda} := \operatorname{Hom}_{\overline{\mathbb{C}}}(V,\mathbb{C}) \,/\, \{\, L \in \operatorname{Hom}_{\overline{\mathbb{C}}}(V,\mathbb{C}) \mid 2\Re L(\Lambda) \subseteq \mathbb{Z}\}.$$

A homomorphism of complex tori $f : V/\Lambda \to V'/\Lambda'$ induces a dual homomorphism

$$\widehat{f} : \widehat{V'/\Lambda'} \to \widehat{V/\Lambda}, \quad L' \mapsto L' \circ f.$$

We always identify complex tori and morphisms between them with their double duals through the evaluation isomorphisms:

$$\begin{tikzcd} V/\Lambda \arrow[r, "f"] \arrow[d, "\mathrm{ev}", "\wr"'] & V'\Lambda' \arrow[d, "\mathrm{ev}", "\wr"'] \\ \widehat{\widehat{V/\Lambda}} \arrow[r, "\widehat{\widehat{f}}"] & \widehat{\widehat{V'\Lambda'}}. \end{tikzcd}$$

Complex sesquilinear forms [3] H on V correspond to homomorphisms ϕ_H from V/Λ to its dual through the maps

$$\left\{\begin{matrix}\text{homomorphisms}\\ \phi\colon V/\Lambda \to \widehat{V/\Lambda}\end{matrix}\right\} \underset{\phi_H := (v \mapsto \frac{1}{2i}H(v,\bullet)) \mapsfrom H}{\overset{\phi \mapsto 2i\,\phi(\bullet)(\bullet)}{\underset{\sim}{\rightleftarrows}}} \left\{\begin{matrix}\text{sesquilinear forms } H\colon V\times V \to \mathbb{C}\\ \text{such that } \Im H(\Lambda,\Lambda) \subseteq \mathbb{Z}\end{matrix}\right\}. \tag{A.1}$$

Lemma A.6 *A dualizing homomorphism* ϕ_H *is an isogeny (resp. an isomorphism) if and only if* H *is non-degenerate (resp.* $\Im H$ *unimodular).*

[2] In our primary reference on Abelian varieties [BL04, 2.4], the dual lattice $\widehat{\Lambda}$ is defined by requiring that the imaginary part $\Im L$ be integer-valued on Λ, instead of $2\Re L$. This results in an isomorphic dual complex torus through multiplication by $2i$. We have avoided the imaginary part convention because then the lattice of $\mathrm{J}^{2k+1} X$ and the map (1.3) have to be multiplied by $2i$, if we want $\mathrm{J}^{2k+1} X$ to be the dual of $\widehat{\mathrm{J}^{2k+1}} X$.

[3] A complex sesquilinear form H on V is $\mathbb{R}$-bilinear, $\mathbb{C}$-linear in the first argument and $\mathbb{C}$-antilinear on the second argument.

A.3 Line bundles on complex tori

The Picard group

$$\mathrm{Pic}(X) = \{L \mid L \text{ holomorphic line bundle on } X\}/\text{isomorphism}$$

is descibed using Čech cohomology and the exponential sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \xrightarrow{\exp(2\pi i \bullet)} \mathcal{O}_X^* \to 0\,.$$

The Čech cocycles of holomorphic transition functions can be described in terms of factors of automorphy, i.e. elements of group cohomology $\mathrm{H}^1(\Lambda, \mathcal{O}_V^*(V))$. In turn, these factors can be described by integral hermitian forms and semicharacters on the multiplicative group $\mathbb{C}_1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Namely, if

$$\mathcal{P}(X) := \left\{ (H, \chi) \,\middle|\, \substack{H: V\times V \to \mathbb{C} \text{ Hermitian form, } \Im H(\Lambda,\Lambda)\subseteq\mathbb{Z} \\ \chi:\Lambda\to\mathbb{C}_1 \text{ semicharacter for } H} \right\}$$

is endowed the group operation $(H_1, \chi_1) \circ (H_2, \chi_2) = (H_1 + H_2, \chi_1 \cdot \chi_2)$, then there is the isomorphism

$$\mathcal{P}(X) \xrightarrow{\sim} \mathrm{H}^1(\Lambda, \mathcal{O}_V^*(V)), \quad (H, \chi) \mapsto a_{(H,\chi)} := \left[(\lambda, v) \in \Lambda \times V \mapsto \chi(\lambda) e^{\pi H(v,\lambda) + \frac{\pi}{2} H(\lambda,\lambda)}\right]$$

that associates to each pair (H, χ) its so-called *canonical factor*. The Appel-Humbert theorem [BL04, 2.2.3] then states that the kernel of the first Chern class $\mathrm{Pic}(X) \xrightarrow{c_1} \mathrm{NS}(X)$ corresponds to the group of characters $\mathrm{Hom}(\Lambda, \mathbb{C}_1) \hookrightarrow \mathcal{P}(X)$. Thus, the Néron-Severi group $\mathrm{NS}(X)$ is isomorphic to the group $\mathcal{H}(X)$ of integral hermitian forms on V. Everything fits together in diagram (A.2), where it is understood that each horizontal sequence of homomorphisms is exact, as well as $\check{\mathrm{H}}^1_{\mathcal{U}}(X, \underline{\mathbb{Z}}) \hookrightarrow \check{\mathrm{H}}^1_{\mathcal{U}}(X, \mathcal{O}_X) \twoheadrightarrow \mathrm{Pic}^0(X)$. However, note that some of the abelian groups in the lower part of the diagram are not $\mathbb{C}$-vector spaces in an obvious way, unless one transports that structure through the vertical isomorphisms.

$$
\begin{tikzcd}
 & \mathrm{H}^1(X,\underline{\mathbb{C}}) \arrow[d, "\wr", "\substack{\text{sheaf}\\ \text{cohomology}}"'] \arrow[dr, two heads, "\substack{\text{Hodge}\\ \text{projection}}"] & & \mathrm{Pic}^0(X) \arrow[r, hook] & \mathrm{Pic}(X) \arrow[r, two heads, "c_1"] \arrow[dd, "\wr"', "\substack{\text{transition}\\ \text{functions}}"] & \mathrm{NS}(X) \arrow[ddr, hook] & & \\
 & \check{\mathrm{H}}^1_{\mathcal{U}}(X,\underline{\mathbb{C}}) \arrow[dr, two heads] & \mathrm{H}^1(X,\mathcal{O}_X) \arrow[d, "\wr"] & & & & & \check{\mathrm{H}}^2_{\mathcal{U}}(X,\underline{\mathbb{C}}) \\
\check{\mathrm{H}}^1_{\mathcal{U}}(X,\underline{\mathbb{Z}}) \arrow[ur, hook] \arrow[rr] & & \check{\mathrm{H}}^1_{\mathcal{U}}(X,\mathcal{O}_X) \arrow[uur, two heads] \arrow[rr, "\exp(2\pi i\bullet)"] & & \check{\mathrm{H}}^1_{\mathcal{U}}(X,\mathcal{O}_X^*) \arrow[rr] & & \check{\mathrm{H}}^2_{\mathcal{U}}(X,\underline{\mathbb{Z}}) \arrow[ur, hook] & \\
 & \mathrm{H}^1(\Lambda,\mathbb{C}) \arrow[uu, "\wr", "\varphi^1_{\mathcal{U}}"'] \arrow[dr, two heads] & & & & & & \mathrm{H}^2(\Lambda,\mathbb{C}) \arrow[uu, "\wr", "\varphi^2_{\mathcal{U}}"'] \\
\mathrm{H}^1(\Lambda,\mathbb{Z}) \arrow[uu, "\wr", "\varphi^1_{\mathcal{U}}"'] \arrow[ur, hook] \arrow[rr] & & \mathrm{H}^1(\Lambda,\mathcal{O}_V(V)) \arrow[uu, "\wr", "\varphi^1_{\mathcal{U}}"'] \arrow[rr, "\exp(2\pi i\bullet)"] & & \mathrm{H}^1(\Lambda,\mathcal{O}_V^*(V)) \arrow[uu, "\wr", "\varphi^1_{\mathcal{U}}"'] \arrow[rr] & & \mathrm{H}^2(\Lambda,\mathbb{Z}) \arrow[uu, "\wr", "\varphi^2_{\mathcal{U}}"'] \arrow[ur, hook] & \\
 & & & & & & & \mathrm{Alt}^2(\Lambda,\mathbb{C}) \arrow[uu, "\wr"] \\
 & \mathrm{Hom}(\Lambda,\mathbb{C}) \arrow[uuu, "\wr"] \arrow[dr, two heads, "F\mapsto \substack{\Re F(\bullet)\\ -\Im F(i\bullet)}"] & & \mathrm{Hom}(\Lambda,\mathbb{C}_1) \arrow[uuuuuu, "\wr"] \arrow[r, hook, "\chi\mapsto(0,\chi)"] & \mathcal{P}(X) \arrow[uu, "\wr", "\substack{a_{(H,\chi)}\\ \uparrow\\ (H,\chi)}"'] \arrow[r, two heads, "(H,\chi)\mapsto H"] & \mathcal{H}(X) \arrow[uuuuuu, "\wr"] \arrow[dr, "\sim", "\substack{H\mapsto\Im H\\ E(i\bullet,\bullet)+iE(\bullet,\bullet)\mapsfrom E}"'] & \mathrm{Alt}^2(\Lambda,\mathbb{Z}) \arrow[uu, "\wr"] \arrow[ur, hook] & \Omega\otimes\overline{\Omega} \arrow[u, hook] \\
\mathrm{Hom}(\Lambda,\mathbb{Z}) \arrow[uuu, "\wr"] \arrow[ur, hook] \arrow[rr, hook] & & \mathrm{Hom}(\Lambda,\mathbb{R}) \arrow[uuu, "\wr"] \arrow[ur, two heads, "\exp(2\pi i\bullet)"'] & & & & \mathrm{Alt}^2(\Lambda,\mathbb{Z})\cap\Omega\otimes\overline{\Omega} \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}
\tag{A.2}
$$

From diagram (A.2) one obtains directly the following moduli interpretation of the dual complex torus.

Lemma A.7 *The Picard group of a complex torus is isomorphic to the dual complex torus through*

$$\widehat{X} \xrightarrow[\sim]{l\mapsto\exp(\pi\, l(\bullet))} \mathrm{Hom}(\Lambda,\mathbb{C}_1) \xrightarrow[\sim]{a_{(0,\bullet)}} \mathrm{Pic}^0 X\,.$$

Definition A.8 *A* theta divisor *for an Abelian variety (X,H) is a divisor Θ on X whose fundamental class $[\Theta] = \mathrm{c}_1(\mathcal{O}_X(\Theta)) \in \mathrm{NS}(X)$ corresponds to the polarization H under the Appel-Humbert correspondence; then we denote the Abelian variety also by the data (X,Θ).*

The cohomology of line bundles on X can be explicitely computed in terms of the canonical factor $a_{(H,\chi)}$; see [BL04, 3]. Two important results are the following.

Theorem A.9 *If (X,H) is an Abelian variety, then there exists an effective line bundle $\mathcal{L}$ on X with $\mathrm{c}_1(\mathcal{L}) \in \mathrm{NS}(X)$ corresponding to H under the Appel-Humbert correspondence. In particular, every Abelian variety admits a theta divisor.*

Theorem A.10 *An Abelian variety (X,Θ) is principally polarized if and only if $\mathrm{h}^0(X,\mathcal{O}(\Theta)) = 1$. In this case, a theta divisor is uniquely determined up to translation by a group element of X.*

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